TA Session 9

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ECON 8040: Macroeconomics I
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Today's Session

- **Problem Set 4** due Friday, Oct. 10 at 11:59p.m.
- **Midterm Exam** on Thursday, Oct. 16, 2:20–3:35p.m. in MLC 275

Problem Set 4

- Due **Friday, October 10 at 11:59 p.m.**
- Four dynamic programming problems
- All use **guess-and-verify** to solve model analytically
- Finish Problem 1, then do other problems
- If stuck, review lecture notes for case w/ log preference
- Useful log properties

$$\log(a \times b) = \log(a) + \log(b) \qquad \log(k^\alpha) = \alpha \times \log(k)$$
$$\log(a/b) = \log(a) - \log(b)$$

Problem 1

(a) Write the Bellman equation

(b) Find value function $v(k)$ and policy function k' by **guess-and-verify**

1. Assume $v(k) = A\frac{k^{1-\sigma}}{1-\sigma}$, find k' using optimality condition
   - It should depend on A, k, and parameters
   - **Hint:** May help simplify math to replace $R \equiv z + 1 - \delta$
2. Replace k' in Bellman and solve for coefficient A
   - A should depend on parameters only, **not** k
3. Replace A in expression from first step to write policy function $g(k)$

(c) Use the policy function for $k' = g(k)$ to find $\frac{k_{t+1}}{k_t}$ and $\frac{c_{t+1}}{c_t}$

Problem 2

(a) Write two things
- Optimality condition for labor supply n using $F(k, k')$
  - You can't write closed-form solution for n, yet
- Bellman equation using $F(k, k')$

(b) Assuming full depreciation and $V(k) = A + B\log(k)$, find k'
- It depends on parameters, B, and n

(c) Write n in terms of parameters and B using results from (a) and (b)

(d) Replace k' and n in guess of $V(k)$ to solve B

(e) Solve for policy functions n, k', and c as function of state k
- Check your work: $\phi = 1 \Rightarrow n = 1$
- Check your work: $n \in [0, 1]$ always (assume $\phi \in [0, 1]$, $\alpha, \beta \in (0, 1)$)

Problem 3

(a) Write planning problem $w(k_0, h_0)$
   - k_0, h_0 is given initial stocks of physical, human capital

(b) Write the planning problem recursively

(c) Assume full depreciation ($\delta = 1$) and use guess-and-verify to solve:
   - $V(k, h)$
   - $k'(k, h)$
   - $h'(k, h)$

Problem 4

(a) Rewrite the problem so $\{k_{t+1}\}_{t=0}^{\infty}$ is only choice variable

(b) Write the problem recursively using two equations
  - $v(k, \theta_L)$
  - $v(k, \theta_H)$
  - You know how state θ_t evolves over time

(c) Solve the Bellman equations using guess-and-verify

(d) Find policy functions $g(k, \theta_L)$ and $g(k, \theta_H)$

(e) Assume $k_0 = 1$ and simulate k_1, k_2, k_3
  - Preference scalar θ_t switches deterministically
  - You know $\theta_0 = \theta_H$, $\theta_1 = \theta_L$, so on

Course Topics

- Static Economy
  - Defining Competitive Equilibrium
    - Production vs. Endowment Economies
    - Government Policy
  - Household Optimality
    - Marginal Rate of Substitution (MRS) & Marginal Rate of Transformation (MRT)
    - Case Study: Consumption, Leisure, & Labor Supply
  - Constant Returns to Scale (CRS) Production Functions
  - Practice: Problem Set 0 #9, Problem Set 1

Course Topics

- Static Economy
- Solow Growth Model
  - Investment, Capital Accumulation, and Economic Growth
  - Proving existence of and convergence to steady state
  - Solving for the steady state
  - Practice: Problem Set 2

Course Topics

- Static Economy
- Solow Growth Model
- Dynamic Economy
  - Develop microfoundations of investment
  - Social Planner's Problem
    - Derive Euler equation & intertemporal optimality
    - Solve finite horizon & infinite horizon planner's problem
  - Household Consumption–Savings Problem
  - Practice: Problem Sets 3 & 4

Course Materials

- Roozbeh's lecture notes and slides
- TA session slides
  - TA5: Problem Set 1 & Problem Set 0 #9
  - TA8: Problem Set 2
  - TA8: Problem Set 3
- Past Midterm Exams on eLC
  - Recommended problems at end of this slide deck

Static Economy

- Defining Competitive Equilibrium
  - Every economy has:
    - Preference-maximizing households
    - Market clearing
  - Endowment economies have no production
  - If consumption good is produced using capital + labor, include profit-maximizing firm that hires inputs from household
  - Government Policy
    - Government's budget must balance (even if you allow it to borrow; every bond issued must be bought)
    - Be sure to incorporate tax into household or firm problem appropriately
    - Transfers do not affect market-clearing; expenditures do!

Static Economy (cont.)

- Structure of CE Definition
  Competitive Equilibrium is prices $\{w, r, \dots\}$, policies $\{\tau, T, \dots\}$, HH allocations $\{c, h, k \dots\}$, and firm allocation $\{Y, N, K, \dots\}$ such that
  1. **Given prices and policy**, household allocation solves preference maximization

     [Define UMP w/ choice variables clearly defined]
  2. **Given prices and policy**, firm allocation maximizes profit

     [Write profit problem w/ choice variables clearly defined]
  3. Government budget equation (if applicable)

     [Tax Revenues = Transfers + Expenditures]
  4. Markets clear

     [Supply equals demand for all goods / services traded in economy]

Static Economy (cont.)

- **Exercise: Problem Set 1, 2(a)**: Define the following competitive equilbrium. Consider an economy with two groups of households. Group one has one unit endowment of good 1 and group 2 has 1 unit endowment of good 2. Think of good 1 and good 2 as consumption when young and consumption when old. Households have the following preferences

$$(1-\beta)u(c_1) + \beta u(c_2)$$

where $0 \leq \beta \leq 1$, and $u(c) = \dfrac{c^{1-\sigma} - 1}{1-\sigma}$ for $\sigma \leq 0$. Let q be the relative price of good 2.

Static Economy (cont.)

- **Exercise: Problem Set 1, 4(a)**: Define competitive equilibrium. Government uses taxes to finance its expenditure g and transfers B. Government taxes labor income at constant rate τ. Suppose government expenditure g equals 20 percent of output, so $g/y = 0.2$. The production function is

$$y = Ak^{\alpha} n^{1-\alpha}$$

where $\alpha = 1/3$. Assume household has the following preferences

$$\phi \log c + (1 - \phi) \log(100 - h)$$

Assume household is endowed with capital $\bar{k}$.

Static Economy (cont.)

- Household Problem (Two Goods Case): Given prices p_1, p_2,

$$\max_{c_1, c_2} U(c_1, c_2) \text{ subject to } p_1 c_1 + p_2 c_2 = W; \lambda$$

  - Necessary FOCs for Interior Solutions

$$\frac{\partial U}{\partial c_1} \equiv U_{c_1} = \lambda p_1 \qquad U_{c_2} = \lambda p_2 \qquad \Longrightarrow \frac{p_2}{p_1} = \frac{U_{c_2}}{U_{c_1}}$$

  - Marginal Rate of Transformation (MRT) equals Marginal Rate of Substitution (MRS)
  - WLOG normalize price of *numeraire* (i.e., set $p_1 = 1$)

Static Economy (cont.)

- Case Study: Labor Supply & Income Taxation
  - Define c_1 as consumption good c (*numeraire*)
  - Define c_2 as leisure ℓ (price is wage w)
  - Government taxes labor income at rate $\tau \geq 0$
- Given wage w and policy τ, household solves

$$\max_{c,\ell} U(c,\ell) \text{ subject to } c + (1-\tau)w\ell = (1-\tau)w; \lambda \qquad 0 \leq \ell \leq 1; \gamma$$

- Necessary first-order condition for interior solutions

$$w(1-\tau) = \frac{U_\ell}{U_c}$$

Static Economy (cont.)

$$w(1-\tau) = \frac{U_\ell}{U_c}$$

- Labor income taxation $\downarrow$ MRT
- Labor income taxation affects MRS through two channels
  - **Substitution effect** $\Longrightarrow \uparrow \ell + \downarrow c \Longrightarrow \downarrow U_\ell + \uparrow U_c$
  - **Income effect** $\Longrightarrow \downarrow \ell + \downarrow c \Longrightarrow \uparrow U_\ell + \uparrow U_c$
- Effect of income taxes on labor supply is theoretically ambiguous!

Static Economy (cont.)

- Constant Returns to Scale (CRS) Production Functions
  - Cobb-Douglas

$$y_t = f(k_t, n_t; A, \alpha) = A k_t^{\alpha} n_t^{1-\alpha}$$

    - $A > 0$: Total factor productivity
    - $\alpha \in [0, 1]$: Output elasticity of capital
  - Constant Elasticity of Subsitition (CES)

$$y_t = f(k_t, n_t; \sigma, \alpha) = \left(\alpha^{\frac{1}{\sigma}} k_t^{\frac{\sigma-1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}} n_t^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

    - $\sigma \in (0, \infty)$: Elasticity of Substitution Capital-Labor
    - $\alpha \in [0, 1]$: Weight of Capital in Production

Static Economy (cont.)

- Useful Properties of CRS Technology
  1. Simplify production sector w/ single **representative firm**
     - Total output depends only on total inputs and *not* the size distribution of firms
     - **Exercise:** Assume economy has J firms. Denote each firm's share of aggregate capital K and aggregate labor N as s_j. Show that

$$\sum_{j=1}^{J} f(s_j K, s_j N) = f(K, N)$$

for Cobb-Douglas, CES, and any CRS production function $f(K, N)$

Static Economy (cont.)

2. Zero-profit condition (under *competitive* input and output markets)
   - W/ CRS technology and perfect competition, firm's total revenue paid *entirely* to production factors
   - Zero profits in equilibrium simplifies national income accounting
   - **Exercise:** Using your expressions for r_t, w_t and profit equation, show that representative firm makes zero profit in equilibrium
3. Factor prices depend only on *ratio* of inputs K/N, not levels K or N
   - Allows for normalization of total inputs w/out loss of generality
   - Common example: Normalize total labor force $N = 1$
   - Helpful for finding steady-state allocations, prices in growth models
   - **You showed this in PS1, Problem 5**

Solow Growth Model

- Model of economic growth + capital accumulation
- Capital evolves according to **law of motion**

$$K_{t+1} = K_t - \delta K_t + sY_t$$

- Key forces in the model:
  - Fixed fraction of capital depreciates $\Longrightarrow \downarrow$ Capital stock
  - Fixed fraction of output saved $\Longrightarrow \uparrow$ Capital stock
- May also assume population, labor productivity grow (exogenously)

Solow Growth Model (cont.)

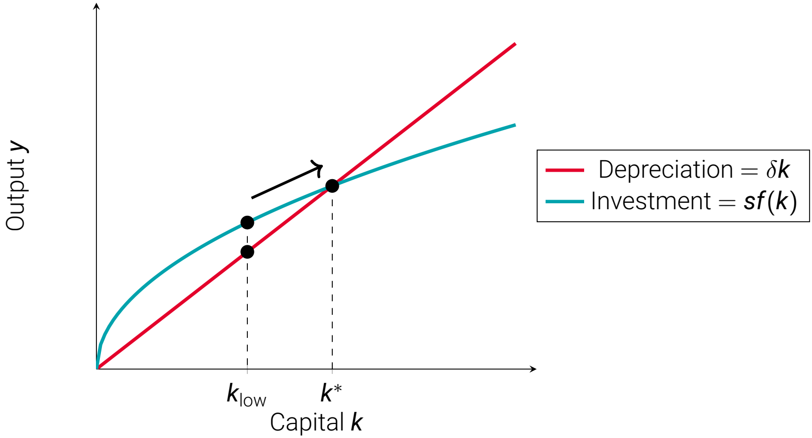

Figure: Converging to Steady State in the Solow Model

Solow Growth Model (cont.)

- Economy converges to long-run *steady state*
  - *Diminishing marginal product* $\Longrightarrow$ Investment grows at slowing rate
  - *Constant depreciation* $\Longrightarrow$ Capital destroyed at constant rate
- Characterizing the steady state
  1. Replace technology into law of motion
  2. Detrend law of motion by any factors that are growing
  3. Impose steady state condition $k^* = k_{t+1} = k_t$ and solve w/ algebra
- **Exercise:** Assume $Y_t = AK_t^{\alpha}[(1+g)N_t]^{1-\alpha}$ and $N_{t+1} = (1+n)N_t$. Show steady state capital is

$$k^* = \left(\frac{sA}{(1+g)(1+n) - 1 + \delta}\right)^{\frac{1}{1-\alpha}}$$

Solow Growth Model (cont.)

$$k^* = \left(\frac{sA}{(1+g)(1+n) - 1 + \delta} \right)^{\frac{1}{1-\alpha}}$$

- Steady state capital per effective worker is
  - $\uparrow$ Savings rate s
  - $\downarrow$ Depreciation rate δ
  - $\uparrow$ TFP A
  - $\uparrow$ Output elasticity of capital α
  - $\downarrow$ Labor-augmenting productivity growth g
  - $\downarrow$ Population growth n

Solow Growth Model (cont.)

- **Example: Problem Set 2, 2(c)** Let $h(k) = (1-\delta)k + sf(k)$, where $s, \delta \in (0,1)$ and $f(k)$ is strictly increasing, concave, and satisfies Inada conditions. Show that the slope of $h(k) < 1$ as k gets large.
- Write a good proof with math *and words*:
  1. (Re)state what you want to show

Solow Growth Model (cont.)

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     We want to show that $\lim_{k\to\infty} h'(k) < 1$.
  2. Write down what you are allowed to assume

Solow Growth Model (cont.)

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- Write a good proof with math *and words*:
  1. (Re)state what you want to show
     We want to show that $\lim_{k\to\infty} h'(k) < 1$.
  2. Write down what you are allowed to assume
     By Inada conditions, $\lim_{k\to\infty} f'(k) = 0$*, and we assume* $\delta \in (0,1)$.
  3. Write steps that *follow from your assumptions* and move towards the statement in (1)

Solow Growth Model (cont.)

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- Write a good proof with math *and words*:
  1. (Re)state what you want to show
     We want to show that $\lim_{k\to\infty} h'(k) < 1$.
  2. Write down what you are allowed to assume
     By Inada conditions, $\lim_{k\to\infty} f'(k) = 0$, *and we assume* $\delta \in (0,1)$.
  3. Write steps that *follow from your assumptions* and move towards the statement in (1)
     The definition of $h(k)$ *implies* $h'(k) = 1 - \delta + sf'(k)$.
     Therefore, $\lim_{k\to\infty} h'(k) = 1 - \delta + s \lim_{k\to\infty} f'(k) = 1 - \delta < 1$. ■

Solow Growth Model (cont.)

- Proofs are structured "linearly" (i.e., each statement must follow from what was stated before)
- When creating proofs, you often write (1), then (3), then (2)
- I call this writing the proof "backwards"
- In previous example,
  - First, write *We want to show* $\lim_{k\to\infty} h'(k) < 1$
  - Second, replace $h'(k)$ and see that you need to show

$$1 - \delta + s \lim_{k\to\infty} f'(k) < 1$$

  - To reach this conclusion, we need to know (or assume) something about δ and $\lim_{k\to\infty} f'(k)$ $\leftarrow$ **This goes into (2)!**

Solow Growth Model (cont.)

- **Exercise: Problem Set 2, 2(f)**: Show that there exists a $\bar{k}$ such that for any $k > \bar{k}$, $k_{t+1} < k_t$.
- **Exercise: Problem Set 2, 2(g)**: Show that the capital stock converges to k^* for any $k_0 \in (0, \bar{k}]$, where we assume $\bar{k} > k^*$.

Dynamic Model

- Solow growth model assumed fixed fraction of output invested
- This section of course introduces microfoundations of investment
- Defining Social Planner's Problem
  - Objective: Maximize present value of lifetime utility
  - Choices: Sequences of consumption and savings
  - Constraint: Feasibility of allocations

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t) \text{ subject to } c_t + k_{t+1} \leq F(k_t, n_t = 1) + (1-\delta)k_t; \lambda_t$$

Dynamic Model (cont.)

- Consumption–Savings Tradeoff Characterized by Euler Equation

$$\begin{aligned} \text{FOC wrt } c_t: &\qquad \lambda_t = \beta^t u'(c_t) \\ \text{FOC wrt } c_{t+1}: &\qquad \lambda_{t+1} = \beta^{t+1} u'(c_{t+1}) \\ \text{FOC wrt } k_{t+1}: &\qquad \lambda_t = \lambda_{t+1}\left[F_k(k_{t+1}, 1) + 1 - \delta\right] \end{aligned}$$

Combining characterizes intertemporal optimality

$$u'(c_t) = \beta[F_k(k_{t+1}, 1) + 1 - \delta]u'(c_{t+1})$$

- Note: You may also obtain Euler equation by replacing c_t with feasibility constraint in objective and writing FOC wrt k_{t+1}

Dynamic Model (cont.)

- Finite time horizon planner's problem $\longleftarrow$ **Backwards induction**
  Exercise: Problem Set 3, 1: Solve optimal sequence of capital $\{k_1, k_2, \ldots, k_T, k_{T+1}\}$ in following planner's problem:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t \log(c_t)$$

$$\text{subject to } c_t + k_{t+1} = Ak_t^\alpha \qquad k_{t+1} \geq 0 \qquad k_0 \text{ given}$$

  1. Replace c_t w/ feasibility constraint

$$\max_{\{k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t \log(Ak_t^\alpha - k_{t+1})$$

Dynamic Model (cont.)

2. Write necessary FOC wrt k_{t+1} to obtain Euler equation

$$\frac{\partial}{\partial k_{t+1}} \cdots + \beta^t \log\left(Ak_t^\alpha - k_{t+1}\right) + \beta^{t+1} \log\left(A{k_{t+1}}^\alpha - k_{t+2}\right) + \cdots = 0$$

$$\underbrace{\frac{1}{Ak_t^\alpha - k_{t+1}}}_{u'(c_t)} = \beta \times \underbrace{\alpha A k_{t+1}^{\alpha-1}}_{F_k+1-\delta\ (\delta=1)} \times \underbrace{\frac{1}{Ak_{t+1}^\alpha - k_{t+2}}}_{u'(c_{t+1})}$$

Dynamic Model (cont.)

3. **Backwards Induction**
   (a) Solve Planner's Problem in final period T

$$\max_{k_{T+1}} \log(Ak_T^\alpha - k_{T+1}) \text{ subject to } k_{T+1} \geq 0 \implies k_{T+1} = 0$$

   (b) Plug $k_{T+1} = 0$ into Euler equation and iterate backwards

$$T-1: \quad \frac{1}{Ak_{T-1}^\alpha - k_T} = \frac{\alpha\beta}{k_T} \implies k_T = \frac{\alpha\beta}{1+\alpha\beta} Ak_{T-1}^\alpha$$

$$T-2: \quad \frac{1}{Ak_{T-2}^\alpha - k_{T-1}} = \frac{\alpha\beta Ak_{T-1}^{\alpha-1}}{Ak_{T-1}^\alpha - k_T} \implies k_{T-1} = \frac{\alpha\beta(1+\alpha\beta)}{1+\alpha\beta+(\alpha\beta)^2} Ak_{T-2}^\alpha$$

   (c) **Exercise:** Recognize pattern and write policy function $k_{t+1}(k_t; t, T, \alpha, \beta, A)$

Dynamic Model (cont.)

- Infinite-horizon problem $\leftarrow$ **Guess-and-Verify Value Function**
  1. Write planner's problem *recursively*

$$
\begin{aligned}
V(k_0) &= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log\left(Ak_t^{\alpha} - k_{t+1}\right) \\
&= \max_{k_1} \left\{ \log(Ak_0^{\alpha} - k_1) + \beta \max_{\{k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} \log(Ak_t^{\alpha} - k_{t+1}) \right\} \\
&= \max_{k_1} \{\log(Ak_0^{\alpha} - k_1) + \beta V(k_1)\}
\end{aligned}
$$

Change of notation: $k \equiv k_t$; $k' \equiv k_{t+1}$

$$
V(k) = \max_{k'} \{\log(Ak^{\alpha} - k') + \beta V(k')\}
$$

Dynamic Model (cont.)

2. Guess the value function: $V(k) = X\log(k) + Y$

$$X\log(k) + Y = \log(Ak^{\alpha} - k') + \beta X\log(k') + \beta Y$$

3. Verify the guess
   (a) Assuming guess is correct, write FOC wrt k'

$$\frac{1}{Ak^{\alpha} - k'} = \frac{\beta X}{k'} \implies k' = \frac{\beta X}{1 + \beta X} Ak^{\alpha}$$

Dynamic Model (cont.)

(b) Replace k' in Bellman and find X

$$\log\left(Ak^\alpha - \frac{\beta X}{1+\beta X}Ak^\alpha\right) + \beta X \log\left(\frac{\beta X}{1+\beta X}Ak^\alpha\right) + \beta Y$$

$$\underbrace{\alpha(1+\beta X)}_{X}\log(k) + \underbrace{\log\left(1-\frac{\beta X}{1+\beta X}\right) + \beta X \log\left(\frac{\beta X}{1+\beta X}\right) + \beta Y}_{Y}$$

$$\therefore X = \alpha + \alpha\beta X \implies X = \frac{\alpha}{1-\alpha\beta}$$

(c) Plug X in k' from FOC to solve policy function

$$k' = \frac{\beta\left(\frac{\alpha}{1-\alpha\beta}\right)}{1+\frac{\alpha\beta}{1-\alpha\beta}} = \alpha\beta A k^\alpha$$

- Planner saves constant share of output, which Solow assumed!

Dynamic Model (cont.)

- We also "microfounded" savings with household problem
  - Objective: Maximize present value of lifetime utility
  - Choices: Sequences of consumption and savings
  - Constraint: **Budget constraint**

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t)$$

subject to $c_t + a_{t+1} = (1+r)a_t + w \qquad a_{T+1} \geq 0$

- Partial equilibrium because prices w, r *exogenous*

Dynamic Model (cont.)

1. Derive **lifetime budget constraint**
   - Combine period budget constraints through savings decisions
     Note: Standard to assume zero return on initial assets a_0

$$c_0 + a_1 = a_0 + w$$

$$c_1 + a_2 = a_1(1+r) + w$$

$$c_1 + a_3 = a_2(1+r) + w$$

   - Write equation relating consumption and a_{T+1} to endowments and a_0

$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} + \underbrace{\frac{a_{T+1}}{(1+r)^T}}_{=0} = a_0 + \sum_{t=0}^{T} \frac{w}{(1+r)^t}; \lambda$$

Why is $a_{T+1} = 0$?

Dynamic Model (cont.)

2. Find initial consumption c_0
   (a) Redefine HH problem as follows:
   Objective: Maximize present value of lifetime utility
   Choices: Sequence of consumption
   Constraint: Lifetime budget constraint
   (b) Write FOCs wrt c_t, c_{t+1} to obtain Euler equation
   (c) Invert $u'(\cdot)$ to write c_{t+1} as function of c_t
   (d) Iterate on equation from (c) to write c_t as function of c_0
   (e) Replace c_t in lifetime budget constraint and solve c_0
3. Solve rest of sequence $\{c_1, \ldots, c_T\}$ using Euler equation
4. Solve savings sequence using period budget constraints

$$a_{t+1} = (1 + r)a_t + w - c_t$$

Practice Problems

- Previous midterm exams are available on eLC
- Time constraint is *binding*. You must work quickly!
- Use past midterms as practice exams:
  - Attempt problems *before* consulting solutions
  - Time yourself – take no more than 75 minutes
  - Then compare your work against solution

Practice Problems (cont.)

- Problems that I recommend trying:
  - Fall 2024, Problem 1
  - Fall 2023 Midterm 2, Problem 1
  - Fall 2023 Midterm 1, Problem 2
  - Fall 2022, Problem 1
  - Fall 2021, Problem 1
  - Fall 2021, Problem 2