TA Session 11

Michael Kotrous
John Munro Godfrey, Sr. Department of Economics
University of Georgia

ECON 8040: Macroeconomics I
November 7, 2025

Revised November 4, 2025

Today's Session

- **Problem Set 5** grades on eLC
- **Problem Set 6** due **Tuesday, November 11** at 5:00 p.m.

Problem Set 5

- Three problems
  1. Recursive planning problems
  2. McCall search model of used car seller
  3. McCall search model of criminal behavior

Problem 1

Given Cobb-Douglas production function, law of capital motion, and three types of preferences $u(c, n)$

log-constant Frisch elasticity	$\log c - \psi \frac{n^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}}$	(1)
log-log	$\phi \log c + (1-\phi)\log(1-n)$	(2)
Cobb-Douglas	$\frac{\left(c^{\phi}(1-n)^{1-\phi}\right)^{1-\sigma}}{1-\sigma}$	(3)

Problem 1 (cont.)

For each preference given

(a) Write recursive planning problem and FOCs, envelope conditions
For log-log preference, DP problem is

$$V(k) = \max_{k',n} \{\phi \log(Ak^\alpha n^{1-\alpha} + (1-\delta)k - k') + (1-\phi)\log(1-n) + \beta V(k')\}$$

subject to $0 \leq k' \leq Ak^\alpha n^{1-\alpha} + (1-\delta)k$, $0 \leq n \leq 1$

- Write FOCs w/ respect to **choices** k', n
- Write envelope condition w/ respect to **state** k
- Notice this DP problem matches PS4 #2

Problem 1 (cont.)

(b) Characterize steady state allocations in terms of k and n

- Write k/n in terms of *only* parameters

  For all 3 preferences, by Euler equation

$$\frac{k}{n} = \left(\frac{\alpha A}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{1}{1-\alpha}}$$

- Write c/n and n in terms of parameters and k/n
  - For all 3 preferences, by feasibility,

$$\frac{c}{n} = A \left(\frac{k}{n} \right)^{\alpha} - \delta \left(\frac{k}{n} \right)$$

- n same for (2) and (3); differs for (1)

Problem 1 (cont.)

- Why is steady state capital per worker same across preferences?

$$u'(c_t) = \beta(1 - \delta + f'(k_{t+1}))u'(c_{t+1})$$

  - We assumed same β, δ, and $f(\cdot)$
  - In steady state, $u'(c_t) = u'(c_{t+1})$, implying

$$f'(k^*) = \frac{1}{\beta} - 1 + \delta \quad \Longrightarrow \quad k^* = (f')^{-1}\left(\frac{1}{\beta} - 1 + \delta\right)$$

  - Steady state capital-per-worker k^* depends on:
    - Discount factor β
    - Depreciation rate δ
    - Technology $f(\cdot)$
    - *Not* the intratemporal tradeoff between c and n

Problem 1 (cont.)

- For $F(K,N) = AK^{\alpha}N^{1-\alpha}$, $f(k) = Ak^{\alpha}$, where $k \equiv K/N$
- Solow economy w/ technology $f(k)$, savings rate s, and full depreciation ($\delta = 1$) attains steady state

$$sf(k^*) = k^* \iff k^* = (sA)^{\frac{1}{1-\alpha}}$$

- PS4 #2: W/ Cobb-Douglas technology, log-log preference, and full depreciation, planner saves fixed output share (as Solow assumed)

$$g(K) = \alpha\beta AK^{\alpha}N^{1-\alpha}$$

$$g(K)/N = \alpha\beta A(K/N)^{\alpha} \xRightarrow{\text{in steady state}} (K/N)^* = (\alpha\beta A)^{\frac{1}{1-\alpha}}$$

Problem 1 (cont.)

- Relaxing full depreciation, Solow steady state

$$k^* = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}$$

- Let $s = \frac{\alpha\beta\delta}{1-\beta(1-\delta)} \in (0,1)$. Then,

$$k^* = \left(\frac{\alpha\beta\delta}{1-\beta(1-\delta)} \times \frac{A}{\delta} \times \frac{1/\beta}{1/\beta}\right)^{\frac{1}{1-\alpha}} = \left(\frac{\alpha A}{\frac{1}{\beta} - 1 + \delta}\right)^{\frac{1}{1-\alpha}}$$

- We've arrived to steady state derived from Euler by previous insight: Planner saves constant share of output each period

Problem 1 (cont.)

- Why is labor supply same for preferences (2) and (3)?
  - Notice that (2) is a special case of (3) with $\sigma = 1$
  - CRRA affects **intertemporal** but not **intratemporal** tradeoffs

$$\underbrace{(1-\alpha)k^{\alpha}n^{-\alpha}}_{\text{MRT}} = \underbrace{\frac{1-\phi}{1-n} \times \frac{c}{\phi}}_{\text{MRS} = U_\ell / U_c}$$

  - Only ϕ (the elasticity of utility wrt consumption) affects MRS
  - Only α affects MRT
  - Labor supply neither *risky* nor *intertemporal*, so σ does *not* affect it

Problem 2

McCall Search Model for Car Seller

(a) Write Bellman equation for choice to accept or reject sale offer

$$V(p) = \max \left\{ p, \beta \int_0^B V(p')dF(p') \right\}$$

(b) Show that Bellman is a contraction mapping

- Discounting: Show $Tv + \beta a = \max\{\dots\} + \beta a \geq T(v + a)$ for $a > 0$
- Monotonicity: Show $Tv \geq Tw$ if $V(p) \geq W(p)$

Problem 2 (cont.)

(c) Optimal policy function characterized by **reservation price** p^*

- Optimal to sell now if $p \geq p^*$ and wait if $p < p^*$
- Use Bellman to write equation that characterizes p^*

$$p^* = \beta \int_0^B V(p')dF(p')$$

- Use assumption that $F(p) \sim U[0, B]$ to find p^*

  1. Split integral from $[0, p^*]$ and $[p^*, B]$ b/c $V(p)$ changes

$$p^* = \beta \left[\int_0^{p^*} \frac{p^*}{B} dp' + \int_{p^*}^{B} \frac{p'}{B} dp' \right]$$

Problem 2 (cont.)

2. Find a quadratic equation in terms of p^*, β, and B

$$0 = \beta(p^*)^2 + (-2B)p^* + \beta B^2$$

3. Apply following form of quadratic equation to solve p^*

$$p^* = \frac{-(-2B) - \sqrt{(-2B)^2 - 4\beta\beta B^2}}{2\beta} = \frac{B(1 - \sqrt{(1 - \beta^2)})}{\beta}$$

- Notice: If $\beta = 1$, then $p^* = B$. **Intuition?**

Problem 2 (cont.)

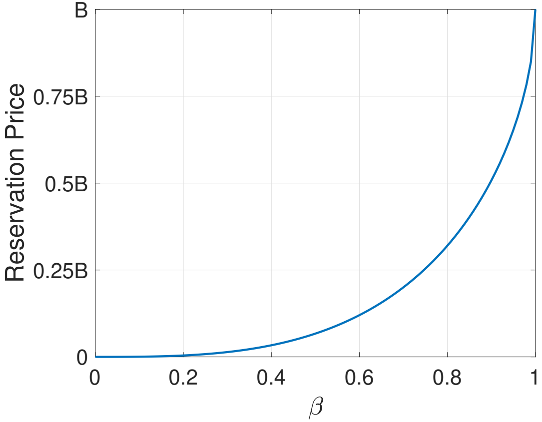

Figure: Reservation price is increasing in patience

Problem 3

Search Model of Criminal Activity

(a-c) Write the payoffs and the Bellman equation

- Payoff for offending:

$$X = p\left[\underbrace{(a-c)\frac{1-\beta^s}{1-\beta}}_{\text{Value Of Incarceration}} + \underbrace{\beta^s E\left[V\left(b\right)\right]}_{\text{Value Once Freed}}\right] + (1-p)\left[\underbrace{a+b+\beta E\left[V(b)\right]}_{\text{Value If Not Caught}}\right]$$

- Payoff for abstaining: $Y = a + \beta E[V(b)]$
- Bellman: $V(b) = \max\{X, Y\}$

(d) Prove / argue that the Bellman has unique solution

- It satisfies discounting and monotonicity

Problem 3 (cont.)

(e) Characterize the policy function to offend / abstain

- Write Bellman imposing assumptions that $a = 0, s = 1$

$$V(b) = \max\{\beta E[V(b)], -pc + p\beta E[V(b)] + (1-p)[b + \beta E[V(b)]]\}$$

- Use Bellman to obtain expression with b^*, $V(b^*)$, and parameters
  For $b = b^*$, $V(b^*) = \beta E[V(b^*)]$. Thus,

$$V(b^*) = -pc + pV(b^*) + (1-p)b^* + (1-p)V(b^*)$$
$$\implies b^* = \frac{p}{1-p} \times c$$

- Threshold at which criminal activity starts *increasing* in probability of being caught and utility cost of incarceration; independent of $F(\cdot)$

Problem 3 (cont.)

(f) Find fraction of free population that is arrested each period

$$\underbrace{p}_{\text{prob. caught}} \times \underbrace{(1 - F(b^*))}_{\text{share who offend}}$$

(g) Find fraction of total population in jail *in steady state*

- "Law of motion" for jail pop: $J_{t+1} = \underbrace{p(1 - F(b^*))}_{\text{share jailed}} \times \underbrace{(1 - J_t)}_{\text{free pop.}}$
- In steady state, $J_t = J_{t+1}$

$$J = \frac{p(1 - F(b^*))}{1 + p(1 - F(b^*))}$$

Problem Set 6 Overview

- Due **Tuesday, November 11** at 5:00 p.m.
- Four dynamic endowment economies
  - Do the problems in order
  - Problem 1 generalizes ADCE example in lecture notes
  - Problem 2-4 tweak endowments or parameters slightly
  - Process for defining/solving ADCE and SMCE consistent throughout
- Bonus problem (worth 10 points) from previous midterm

Problem 1

a) Define Arrow-Debreu CE
   - One budget constraint
   - Infinite market clearing conditions

b) Derive Euler equation
   - First-order conditions with respect to two choice variables

c) Solve for Pareto efficient allocations
   - I recommend using the hint to define $\alpha \equiv \frac{\alpha_1}{\alpha_2}$ when solving
   - **No prices**, only allocations and Pareto weights

Problem 1 (cont.)

d) Negishi Method
  - Find Arrow-Debreu price p_t
    - Use planner's FOC and "shadow price" / Lagrange multiplier λ_t
  - Find ratio of Pareto weights α such that transfers $t^i(\alpha_1, \alpha_2) = 0$
    - Plug price, planner allocations, and endowments in AD budget
  - Find CE allocations by replacing α in planner allocations

e) Find Pareto weights and transfers that yield equal allocations
  - Use c_t^1, c_t^2 that are given, "shadow price" λ_t, and endowments
  - **Second Welfare Theorem:** Any Pareto efficient allocation can be CE allocation w/ some lump-sum transfer in first period!

Problem 1 (cont.)

f) How do prices, growth rate of consumption depend on γ, σ?

- Find $g_t \equiv \dfrac{c^1_{t+1} + c^2_{t+1}}{c^1_t + c^2_t}$
- Evaluate four derivatives

$$\frac{\partial g_t}{\partial \gamma} \qquad \frac{\partial p_t}{\partial \gamma}$$

$$\frac{\partial g_t}{\partial \sigma} \qquad \frac{\partial p_t}{\partial \sigma}$$

Problem 1 (cont.)

g) Sequential Markets CE
  - Define SMCE
    - Budget constraint for **every period**
    - Markets clear in every period
  - Find interest rate. How does it depend on γ, σ?
    - Derive Euler equation
    - Use fact that SMCE allocations = ADCE allocations to find $1 + i_{t+1}$
    - Evaluate two derivatives

$$\frac{\partial(1+i_{t+1})}{\partial\gamma} \qquad \frac{\partial(1+i_{t+1})}{\partial\sigma}$$

Problem 2

a) Find allocations and Arrow-Debreu price w/ different endowments
   - Process for solving similar to lecture example, Problem 1
   - Be careful writing market-clearing conditions
   - Consider $t = 0, 2, \ldots$ and $t = 1, 3, 5, \ldots$ cases separately

b) Sequential Markets CE
   - Define SMCE
     - Budget constraint for **every period**
     - Markets clear every period
   - Find interest rate $1 + i_{t+1}$
     - Derive Euler equation
     - Use fact that SMCE allocations $=$ ADCE allocations from a)

Problem 3

- Both types receive equal endowments every period
- Find equilibrium allocations and prices
- Do allocations differ from endowments? Why or why not?

Problem 4

Households $i \in \{1, 2\}$ differ in discount factor β_i

a) Define ADCE and derive Euler equation
b) Solve planner problem for arbitrary Pareto weights α_1, α_2
   - Easier to solve in terms of $\alpha \equiv \frac{\alpha_2}{\alpha_1}$
c) Negishi Method to solve for CE allocations and prices
d) Evaluate $\lim_{t \to \infty} \frac{c_t^1}{c_t^2}$
e) Provide intution. Do your results match your *ex ante* belief about which type should consume more in the long-run?