TA Session 10

Michael Kotrous
John Munro Godfrey, Sr. Department of Economics
University of Georgia

ECON 8040: Macroeconomics I
October 24, 2025

Revised October 22, 2025

Today's Session

- **Midterm Exam** grades on eLC
- **Problem Set 4** grades on eLC
- **Problem Set 5** due Thursday, October 30 at 11:59 p.m.

Midterm Exam Point Allocations

1. (62 points) Dynamic Programming Problem
   (a) (5 points) Write necessary optimality condition
   (b) (8 points) Define planner's problem in terms of only capital sequence
   (c) (12 points) Write problem recursively
   (d) (16 points) Solve Bellman by guess-and-verify
   (e) (16 points) Write policy functions
   (f) (5 points) Show allocations satisfy optimality condition
2. (38 points) Consumption-Savings w/ Labor Supply & Retirement
   (a) (22 points) Solve allocations
   (b) (16 points) Solve w/ binding borrowing constraint

Midterm Exam Grade Distribution

	Problem 1	Problem 2	Overall
75 pctl.	48	26	79
Median	46	17	63
25 pctl.	37	12	46
Average	41	19	61
Out of ...	62	38	100

Problem 1

- Similar to PS4, #4 but w/ deterministic TFP shock

(a) "Necessary optimality condition" is **Euler equation**

(b) Use **feasibility** to write consumption as function of savings
  - Non-negativity constraint still holds
  - Upper-bound on savings ensures consumption remains feasible

(c) Write Bellman equation(s)
  - **Control variables** are decisions made by agent
  - **State variables** characterize current condition of dynamic system
    - Fixed at start of period
    - May vary across agents or over time
    - May evolve deterministically, stochastically, or by agent's choices

Problem 1 (cont.)

(d) Solve by **guess and verify**

- Guess $V(k, z_i) = A_i \log(k) + D_i$ for $i \in \{L, H\}$
- Characterize optimal k' assuming guess is correct
- Replace k' in Bellman and solve A_i

(e) Write **policy functions**

- Replace A_L, A_H in equation characterizing k'
- Use feasibility to write consumption function

(f) Use policy functions to verify Euler equation holds

Problem 2

(a) Solve "standard" two-period HH problem. Choose your adventure:

- **Option #1**
  - Replace budget contraints in objective
  - Write FOC for h to find h^*
  - Write FOC for a to find a^*
  - Use budget constraints to find c_0^*, c_1^*
- **Option #2**
  - Derive lifetime budget constraint
  - Write FOC for h to find h^*
  - Write FOCs for c_0, c_1 to derive Euler
  - Use Euler and lifetime budget constraint to find c_0^*, c_1^*
  - Use budget constraint to find a^*

Problem 2 (cont.)

- Comparative statics of retirement benefit
  - $\uparrow$ Old-age consumption
  - $\uparrow$ Young-age consumption
    - If benefit large, young borrow against future benefit to $\uparrow$ consumption
  - $\downarrow$ Private savings (crowd-out effect)
  - No labor-supply distortion
    - Not general: Assumed preferences eliminate income effects
    - Abstract from labor-income taxation financing pensions

Problem 2 (cont.)

(b) Add borrowing constraint ($a \geq 0$)

- From (a), HH wants to borrow ($a < 0$) only if benefit large
- If borrowing desired but infeasible $\implies a^* = 0$
- Labor supply unaffected by borrowing constraint
  - No income effects $\implies$ MRS unchanged
- Use budget constraints to solve for c_0^*, c_1^*
- **Intuition:** credit-constrained HH consumes “hand-to-mouth”

Problem Set 4

- Four dynamic programming problems
- All use **guess-and-verify** to solve model analytically
  1. Write dynamic programming problem
  2. Guess functional form of value function
  3. Assuming guess is correct, write optimality condition for k'
  4. Replace k' in Bellman and verify guess
  5. Find policy function $k' \equiv g(k)$

Problem 1

(a) Write the Bellman equation

$$V(k) = \max_{k'} \left\{ \frac{(zk + (1-\delta)k - k')^{1-\sigma}}{1-\sigma} + \beta V(k') \right\}$$

subject to $0 \leq k' \leq zk + (1-\delta)k$

(b) Find value function $v(k)$ and policy function k' by **guess-and-verify**

1. Assume $v(k) = A\frac{k^{1-\sigma}}{1-\sigma}$, find k' using optimality condition

$$k' = \frac{Rk}{1 + (\beta A)^{\frac{-1}{\sigma}}}$$

where $R = z + 1 - \delta$

Problem 1 (cont.)

2. Replace k' in Bellman and solve A and value function $V(k)$

$$A = R^{1-\sigma}\beta A(1 + (\beta A)^{\frac{-1}{\sigma}})^{\sigma}$$

$$A = \frac{R^{1-\sigma}}{\left(1 - R^{\frac{1-\sigma}{\sigma}}\beta^{\frac{1}{\sigma}}\right)^{\sigma}}$$

$$\implies V(k) = \left(\frac{R^{1-\sigma}}{\left(1 - R^{\frac{1-\sigma}{\sigma}}\beta^{\frac{1}{\sigma}}\right)^{\sigma}}\right)\frac{k^{1-\sigma}}{1-\sigma}$$

3. Replace A in expression for k' to write policy function $g(k)$

$$g(k) = (\beta R)^{\frac{1}{\sigma}} k$$

Problem 1 (cont.)

(c) Use the policy function for $k' = g(k)$ to find $\frac{k_{t+1}}{k_t}$ and $\frac{c_{t+1}}{c_t}$

- First, notice

$$\frac{c_{t+1}}{c_t} = \frac{Rk_{t+1} - g(k_{t+1})}{Rk_t - g(k_t)} = \frac{(R - (\beta R)^{\frac{1}{\sigma}})k_{t+1}}{(R - (\beta R)^{\frac{1}{\sigma}})k_t} = \frac{k_{t+1}}{k_t}$$

- Second, notice

$$\frac{k_{t+1}}{k_t} = \frac{g(k_t)}{k_t} = (\beta R)^{\frac{1}{\sigma}}$$

- Therefore, economy grows if and only if

$$\beta R = \beta(z + 1 - \delta) > 1$$

Problem 2

(a) Write two things

- Optimality condition for labor supply n using $F(k, k')$

$$\frac{1-\phi}{1-n} = \frac{\phi(1-\alpha)z\left(\frac{k}{n}\right)^{\alpha}}{zk^{\alpha}n^{1-\alpha} + (1-\delta)k - k'}$$

- Bellman equation using $F(k, k')$

$$V(k) = \max_{k'} \{F(k, k') + \beta V(k')\}$$

subject to $0 \leq k' \leq zk^{\alpha}n^{1-\alpha} + (1-\delta)k, \qquad 0 \leq n \leq 1$

Problem 2 (cont.)

(b) Assuming full depreciation and $V(k) = A + B\log(k)$, find k'

$$k' = \frac{\beta B}{\phi + \beta B} \cdot zk^{\alpha}n^{1-\alpha}$$

(c) Write n in terms of parameters and B using results from (a) and (b)

$$n = \frac{(1-\alpha)(\phi + \beta B)}{1 - \phi + (1-\alpha)(\phi + \beta B)}$$

(d) Replace k' and n in guess of $V(k)$ to solve B

$$B = \frac{\alpha\phi}{1 - \alpha\beta}$$

Problem 2 (cont.)

(e) Solve for policy functions n, k', and c as function of state k

- First, find optimal labor supply

$$n = \frac{(1-\alpha)\phi}{(1-\alpha\beta)(1-\phi) + (1-\alpha)\phi}$$

  - Optimal labor supply invariant in k
  - Sanity check #1: $\phi = 1 \implies n = 1$ and $\phi = 0 \implies n = 0$
  - Sanity check #2: $\alpha = 1 \implies n = 0$ (MPL is zero, so no need to work)
  - Sanity check #3: $n \in [0, 1]$ for all $(\phi, \alpha, \beta) \in [0, 1]$

- Second, find optimal savings

$$g(k) = \alpha\beta z k^{\alpha} \left(\frac{(1-\alpha)\phi}{(1-\alpha\beta)(1-\phi) + (1-\alpha)\phi} \right)^{1-\alpha}$$

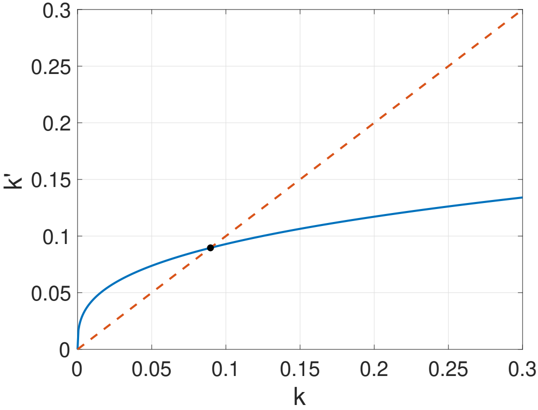
Problem 2 (cont.)

- Third, use n, $g(k)$ to find optimal consumption

$$
\begin{aligned}
c(k) &= zk^{\alpha}n^{1-\alpha} - g(k) \\
&= (1-\alpha\beta)zk^{\alpha}\left(\frac{(1-\alpha)\phi}{(1-\alpha\beta)(1-\phi)+(1-\alpha)\phi}\right)^{1-\alpha}
\end{aligned}
$$

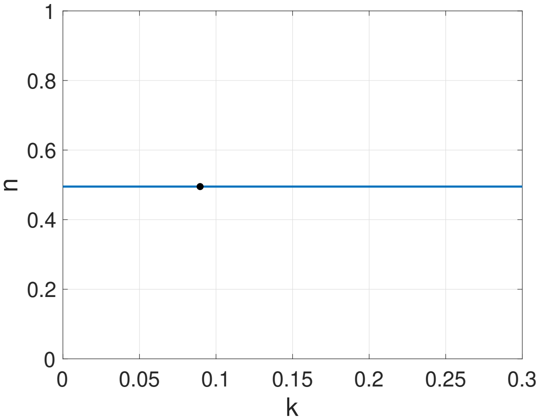
Problem 2 (cont.)

Assume $\alpha = 1/3$, $z = 1$, $\beta = 0.96$, and $\phi = 1/2$



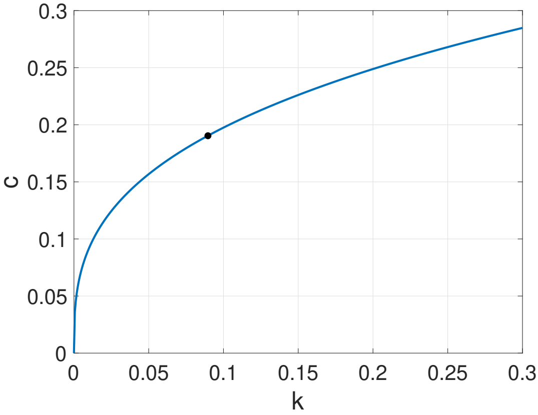
(a) Capital policy function

Problem 2 (cont.)



(b) Labor supply policy function

Problem 2 (cont.)



(c) Consumption policy function

Problem 3

(a) Write planning problem $w(k_0, h_0)$

$$w(k_0, h_0) = \max_{\{c_t, k_{t+1}, h_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log c_t$$

subject to

$$c_t + k_{t+1} + h_{t+1} = z k_t^{\alpha} h_t^{1-\alpha} + (1-\delta)(k_t + h_t)$$
$$c_t, k_{t+1}, h_{t+1} \geq 0$$
$$k_0, h_0 \text{ given}$$

Problem 3 (cont.)

(b) Write the planning problem recursively

$$V(k,h) = \max_{k',h'} \left\{ \log\left(zk^{\alpha}h^{1-\alpha} + (1-\delta)(k+h) - k' - h' \right) + \beta V(k',h') \right\}$$

subject to $0 \leq k', h', k' + h' \leq zk^{\alpha}h^{1-\alpha} + (1-\delta)(k+h)$

(c) Assume full depreciation ($\delta = 1$) and use guess-and-verify to solve:

- Guess $A + D\log(k) + E\log(h)$
- Write equations that characterize optimal k', h'

$$k' = \frac{\beta D(zk^{\alpha}h^{1-\alpha} - h')}{1+\beta D} \qquad h' = \frac{\beta E(zk^{\alpha}h^{1-\alpha} - k')}{1+\beta E}$$

$$k' = \frac{\beta D}{1+\beta D+\beta E} \cdot zk^{\alpha}h^{1-\alpha} \qquad h' = \frac{\beta E}{1+\beta D+\beta E} \cdot zk^{\alpha}h^{1-\alpha}$$

Problem 3 (cont.)

- Replace k', h' in Bellman to solve D and E

$$D = \frac{\alpha}{1-\beta} \qquad E = \frac{1-\alpha}{1-\beta}$$

- Use D and E to write policy functions

$$k'(k,h) = \alpha\beta z k^{\alpha} h^{1-\alpha} \qquad h'(k,h) = (1-\alpha)\beta z k^{\alpha} h^{1-\alpha}$$

Problem 4

(a) Rewrite the problem so $\{k_{t+1}\}_{t=0}^{\infty}$ is only choice variable

- Replace consumption using feasibility constraint

$$w(\bar{k}_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \beta^t \theta_t \log(k_t - k_{t+1})$$

subject to

$$0 \leq k_{t+1} \leq k_t$$
$$\bar{k}_0 \text{ given}$$
$$\theta_t = \begin{cases} \theta_H & t \text{ is even} \\ \theta_L & t \text{ is odd} \end{cases}$$

Problem 4 (cont.)

(b) Write the problem recursively using two equations

$$V(k,\theta_L) = \max_{k'} \left\{\theta_L \log(k-k') + \beta V(k,\theta_H)\right\}$$
$$V(k,\theta_H) = \max_{k'} \left\{\theta_H \log(k-k') + \beta V(k,\theta_L)\right\}$$

Both subject to $0 \leq k' \leq k$

(c) Solve the Bellman equations using guess-and-verify

- Assume following guesses for two Bellmans

t is even: $V(k,\theta_H) = E + H\log(k)$

t is odd: $V(k,\theta_L) = A + D\log(k)$

Problem 4 (cont.)

- Write FOCs for k' in even/odd periods

$$k' = \begin{cases} \frac{\beta D}{\theta_H + \beta D} \cdot k & t \text{ is even} \\ \frac{\beta H}{\theta_L + \beta H} \cdot k & t \text{ is odd} \end{cases}$$

- Replace k' in Bellmans to solve for coefficients D and H

$$D = \frac{\theta_L + \beta\theta_H}{1 - \beta^2} \qquad H = \frac{\theta_H + \beta\theta_L}{1 - \beta^2}$$

Problem 4 (cont.)

(d) Find policy functions $g(k, \theta_L)$ and $g(k, \theta_H)$

$$g(k, \theta_H) = \frac{\beta(\theta_L + \beta\theta_H)}{\theta_H + \beta\theta_L} \cdot k$$

$$g(k, \theta_L) = \frac{\beta(\theta_H + \beta\theta_L)}{\theta_L + \beta\theta_H} \cdot k$$

Problem 4 (cont.)

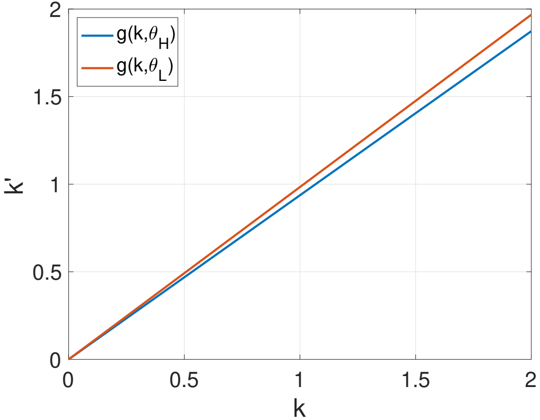


Figure: Policy functions for 4(d)

Problem 4 (cont.)

(e) Assume $k_0 = 1$ and simulate k_1, k_2, k_3

- $k_1 = g(1, \theta_H) = \frac{\beta(\theta_L + \beta\theta_H)}{\theta_H + \beta\theta_L}$
- $k_2 = g(k_1, \theta_L) = \beta^2$
- and so on

Problem 4 (cont.)

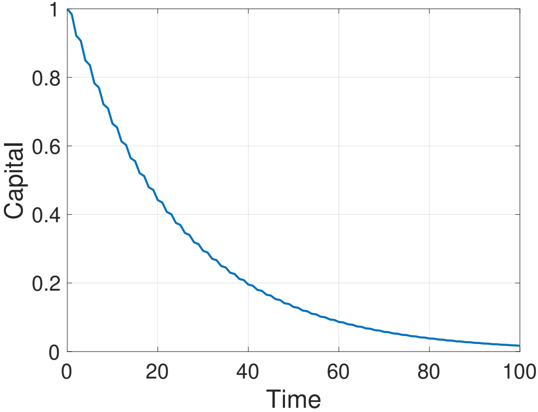


Figure: Simulation of capital for 100 periods

Problem Set 5

- Due **Thursday, October 30** at 11:59 p.m.
- Three problems
  1. Recursive planning problems
  2. McCall search model of used car seller
  3. McCall search model of criminal behavior

Problem 1

Given Cobb-Douglas production function, law of capital motion, and three types of preferences $u(c, 1-n)$

log-constant Frisch elasticity	$\log c - \psi \frac{n^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}}$
log-log	$\phi \log c + (1-\phi)\log(1-n)$
Cobb-Douglas	$\frac{\left(c^{\phi}(1-n)^{1-\phi}\right)^{1-\sigma}}{1-\sigma}$

Problem 1 (cont.)

For each preference

(a) Write recursive planning problem and FOCs, envelope conditions
   - FOCs with respect to *all* choice variables

(b) Characterize *steady state* allocations in terms of k and n
   - Write k/n in terms of *only* parameters
   - Write c/n and n in terms of parameters and k/n

Problem 2

McCall Search Model for Used Car Seller

(a) Write Bellman equation for choice to accept or reject sale offer
   - Depends on offer p and discounted expected value of future offer

(b) Show that Bellman is a contraction mapping
   - Discounting: Show $Tv + \beta a = \max\{\dots\} + \beta a \geq T(v + a)$ for $a > 0$
   - Monotonicity: Assume $V(p) \geq W(p)$. Show $Tv \geq Tw$.

Problem 2 (cont.)

(c) Optimal policy function characterized by **reservation price** p^*

- Optimal to sell now if $p \geq p^*$ and wait if $p < p^*$
- Use Bellman to write equation that characterizes p^*
- Use assumption that $F(p) \sim U[0, B]$ to find p^*
  1. Split integral from $[0, p^*]$ and $[p^*, B]$ b/c Bellman changes
  2. Find a quadratic equation in terms of p^*, β, and B

$$0 = a(p^*)^2 + bp^* + c$$

  3. Apply following form of quadratic equation to solve p^*

$$p^* = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- **Hint:** If $\beta = 1$, then $p^* = B$. **Intuition?**

Problem 3

Search Model of Criminal Activity

(a-c) Write the payoffs and the Bellman equation

- Payoff for offending is an expectation
- W/ prob. p, offender is incarcerated for s periods and then freed
- W/ prob. $1 - p$, offender is not caught and free in next period

(d) Prove / argue that the Bellman has unique solution

(e) Find payoff b^* above which agent commits crime

- Rewrite Bellman imposing assumptions $a = 0, s = 1$
- Use Bellman to obtain expression with b^*, $V(b^*)$, and parameters
- Use expression to write b^* in terms of parameters

Problem 3 (cont.)

(f) Find fraction of free population that is arrested each period
  - Use $F(\cdot)$ to characterize share of population that offends

(g) Find fraction of total population in jail *in steady state*
  - Use answer from (f) to write “law of motion” for jailed population
  - In steady state, # jailed now (J_t) equals # jailed tomorrow (J_{t+1})