Sequential Markets Competitive Equilibrium

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September 12, 2023

Consider a simple endowment economy with two types of households. Households are equivalent in preferences but differ in their endowments. There is no uncertainty in endowments, and households have no initial assets. The key distinction from Arrow-Debreu economy is that trade occurs every period. In sequential markets, households are free to lend and borrow consumption good through one-period, risk-free Arrow securities, subject to the asset market balancing equation and borrowing constraint that prevents Ponzi schemes. This document presents the algebra of solving for sequential markets competitive equilibrium in two-period and infinite horizon economies. The results establish that allocations of consumption good are equivalent to the allocations in the Arrow-Debreu economy with the same preferences and endowments.

1 SMCE in two-period economy

In this exercise, assume households live only two periods ($t = 0, 1$), have log preferences, and are given the following endowments: $e_t^1 = (2, 0)$ and $e_t^2 = (0, 2)$.

A *sequential market competitive equilibrium* (SMCE) is rate of return r and allocations $(c_0^1, c_1^1, s_1^1, c_0^2, c_1^2, s_1^2)$ such that

A. Given rate of return r, allocation by household of type $i = 1, 2$ solves its utility maximization problem:

$$\max_{c_0^i, c_1^i, s_1^i} \log(c_0^i) + \beta \log(c_1^i)$$

subject to

$$\begin{aligned} c_0^i + s_1^i &= e_0^i \\ c_1^i &= (1 + r)s_1^i + e_1^i \\ s_0^i &= 0 && \text{i.e., no initial assets} \end{aligned}$$

Notice that households consume all of their wealth in the final period ($t = 1$). This is because household has no preference for bequest.

B. Markets clear

i. Consumption:

$$c_0^1 + c_0^2 = 2$$
$$c_1^1 + c_1^2 = 2$$

ii. Asset: $s_1^1 + s_1^2 = 0$. Households lend among each other in this economy. Because every borrower must have a lender, net assets among all households equals zero.

1.1 Derive lifetime budget constraint

In sequential markets, household faces a budget constraint for every period (so in this case has two budget constraints). We can use savings s_1^i to rewrite as one "lifetime budget constraint."

$$c_0^i + s_1^i = e_0^i$$
$$c_1^i = (1+r)s_1^i + e_1^i \Rightarrow s_1^i = \frac{c_1^i - e_1^1}{1+r}$$
$$c_0^i + \frac{c_1^i - e_1^1}{1+r} = e_0^i$$
$$c_0^i + \frac{c_1^1}{1+r} = e_0^i + \frac{e_1^i}{1+r} \tag{1}$$

Equation (1) tells us that the present value of lifetime consumption equals the present value of all endowments.

1.2 Solve for equilibrium gross return

Using lifetime budget constraint, we can rewrite household i's optimization problem as

$$\max_{c_0^i, c_1^i} \log(c_0^i) + \beta \log(c_1^i)$$

subject to $c_0^i + \frac{c_1^1}{1+r} = e_0^i + \frac{e_1^i}{1+r}; \lambda$.

Household's first-order conditions are:

wrt c_0^i: $$\frac{1}{c_0^i} = \lambda$$

wrt c_0^i: $$\frac{\beta}{c_1^i} = \frac{\lambda}{1+r}$$

Rearranging and combining, we have household i's Euler equation

$$c_1^i = \beta(1+r)c_0^i \tag{2}$$

Using the Euler equation and market clearing conditions,

$$\begin{aligned} c_1^1 + c_1^2 &= \beta(1+r)c_0^1 + \beta(1+r)c_1^2 \\ &= \beta(1+r)(c_0^i + c_0^2) \\ 2 &= 2\beta(1+r) \\ 1+r &= \frac{1}{\beta} \end{aligned} \tag{3}$$

Equation (3) gives the gross return in equilibrium.

1.3 Solve for equilibrium allocations

Use Euler, budget constraint, and endowments to solve for equilibrium allocation.

Replacing for gross return in (2), we have that

$$\begin{aligned} c_1^i &= \beta(1+r)c_0^i \\ c_1^i &= c_0^i \end{aligned}$$

That is, both household's "smooth" consumption in equilibrium.

We can now use the lifetime budget constraint, (1),to solve for allocations

$$\begin{aligned} c_0^i + \beta c_0^i &= e_0^i + \beta e_1^i \\ c_0^1 = c_1^1 &= \frac{2}{1+\beta} \\ c_0^2 = c_1^2 &= \frac{2\beta}{1+\beta} \end{aligned}$$

Use first period's budget constraint to solve for equilibrium lending/borrowing:

$$\begin{aligned} s_1^i &= e_0^i - c_o^i \\ s_1^1 &= 2 - \frac{2}{1+\beta} = \frac{2\beta}{1+\beta} \\ s_1^2 &= \frac{-2\beta}{1+\beta} \end{aligned}$$

In period 0, the "rich" type lends to the "poor" type. In period 1, "poor" type pays off the loan.

Exercise. Verify that equilibrium rate of return and allocations satisfy all budget constraints and market clearing conditions.

2 SMCE in infinite-horizon economy

We now assume that households are infinitely-lived, have log preferences over consumption, and are given the following endowments. For $i = 1$,

$$e_t^1 = (2, 0, 2, 0, 2, 0, \dots)$$

For $i = 2$,

$$e_t^2 = (0, 2, 0, 2, 0, 2, \dots)$$

We will see that the procedure for solving the simple two-period model extends to solving the infinite-horizon model. Each period, households lend and save by trading one-period, risk-free Arrow securities, denoted a_{t+1}. In period t, households trade a_{t+1}, which is an agreement to deliver a_{t+1} units of consumption good in period $t+1$. So when $a_{t+1} < 0$, household is borrowing. When $a_{t+1} > 0$, household is saving/lending.

SMCE is rate of return $\{r_t\}_{t=0}^{\infty}$ and allocations $\{c_t^1, a_{t+1}^1, c_t^2, a_{t+1}^2\}_{t=0}^{\infty}$ such that

(A) Given rate of return $\{r_t\}_{t=0}^{\infty}$, household of type $i = 1, 2$ makes allocation that solves its utility maximization problem:

$$\max_{\{c_t^i, a_{t+1}^i\}} \sum_{t=0}^{\infty} \beta^t \log(c_t^i)$$

subject to

$$\begin{aligned} c_t^i + a_{t+1}^i &= (1 + r_t)a_t^i + e_t^i \text{ for all } t \\ c_t^i &\geq 0 \text{ for all } t \\ a_{t+1}^i &\geq -\bar{A} \text{ for all } t, \text{ some } \bar{A} > 0 \\ a_0^i &= 0 \text{ i.e., no initial assets} \end{aligned}$$

(B) Markets clear

i. Consumption: $c_t^1 + c_t^2 = 2$ for all t

ii. Assets/Securities: $a_{t+1}^1 + a_{t+1}^2 = 0$ for all t

2.1 Derive lifetime budget constraint

Like we did in the two-period example, we can rewrite budget constraints for $t = 1, 2, \dots$ as a function of previous period's savings decision. We can then iteratively replace for savings in the initial period's budget constraint to derive the lifetime budget constraint.

$$c_0^i = e_0^i - a_1^i$$
$$c_1^i = (1+r_1)a_1^i + e_1^i - a_2^i \Rightarrow a_1^i = \frac{c_1^i - e_1^i + a_2^i}{1+r_1}$$
$$c_2^i = (1+r_2)a_2^i + e_2^i - a_3^i \Rightarrow a_2^i = \frac{c_2^i - e_2^i + a_3^i}{1+r_2}$$
$$\Rightarrow c_0^i = e_0^i - \left(\frac{c_1^i - e_1^i + \left(\frac{c_2^i - e_2^i + a_3^i}{1+r_2}\right)}{1+r_1} \right)$$
$$c_0^i + \frac{c_1^i}{1+r_1} + \frac{c_2^i}{(1+r_1)(1+r_2)} + \frac{a_3^i}{(1+r_1)(1+r_2)} = e_0^i + \frac{e_1^i}{1+r_1} + \frac{e_2^i}{(1+r_1)(1+r_2)}$$

Extending this math for T periods (and defining $r_0 = 0$) we have:

$$\sum_{t=0}^{T} \frac{c_t^i}{\Pi_{j=0}^t (1+r_j)} + \frac{a_{T+1}^i}{\Pi_{j=0}^T (1+r_j)} = \sum_{t=0}^{T} \frac{e_t^i}{\Pi_{j=0}^t (1+r_j)}$$

We are going to impose the following condition: $\lim_{T\to\infty} \frac{a_{T+1}^i}{\Pi_{j=0}^T (1+r_j)} = 0$. This condition states that the long-run value of household's savings converges to zero. This is related to the observation in the two-period model that households do not save in the final period of life.

Hence, the infinitely-lived household's lifetime budget constraint is given by:

$$\sum_{t=0}^{\infty} \frac{c_t^i}{\Pi_{j=0}^t (1+r_j)} = \sum_{t=0}^{\infty} \frac{e_t^i}{\Pi_{j=0}^t (1+r_j)} \tag{4}$$

Recall that $r_0 = 0$ in order for this result to hold. A general convention in macro is to assume that initial assets a_0^i do not earn a rate of return, even in the case where $a_0^i \neq 0$.

2.2 Solve for equilibrium gross return

Using the lifetime budget constraint in (4), we rewrite household's utility maximization problem as

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t^i)$$

subject to

$$\sum_{t=0}^{\infty} \frac{c_t^i}{\Pi_{j=0}^t (1+r_j)} = \sum_{t=0}^{\infty} \frac{e_t^i}{\Pi_{j=0}^t (1+r_j)}; \lambda$$

where $r_0 = 0$. The nonnegativity constraint on consumption does not bind with log preferences, so household's first order conditions are:

$$\text{wrt } c_t^i: \quad \frac{\beta^t}{c_t^i} = \frac{\lambda}{\Pi_{j=0}^t(1+r_j)}$$
$$\text{wrt } c_{t+1}^i: \quad \frac{\beta^{t+1}}{c_{t+1}^i} = \frac{\lambda}{\Pi_{j=0}^{t+1}(1+r_j)}$$

This implies

$$\frac{\beta^t \Pi_{j=0}^t(1+r_j)}{c_t^i} = \frac{\beta^{t+1}\Pi_{j=0}^{t+1}(1+r_j)}{c_{t+1}^i}$$
$$c_{t+1}^i = \beta(1+r_{t+1})c_t^i \tag{5}$$

Equation (5) is the household's Euler equation. By the market clearing conditions,

$$c_{t+1}^1 + c_{t+1}^2 = \beta(1+r_{t+1})c_t^1 + \beta(1+r_{t+1})c_t^2$$
$$2 = \beta(1+r_{t+1})(c_t^1 + c_t^2)$$
$$2 = 2\beta(1+r_{t+1})$$
$$1 + r_{t+1} = \frac{1}{\beta}$$

Because this holds for all t, the equilibrium gross return is a constant, so we can drop the t subscript:

$$1 + r = \frac{1}{\beta} \tag{6}$$

2.3 Solve for equilibrium allocations

Replacing for the gross return in the Euler equation (5), we see that $c_{t+1}^i = c_t^i$, so we again see "consumption smoothing" behavior from household's in this economy.

Consumption and the gross return (6) is constant, so we can simplify the lifetime budget constraint (4):

$$c_t^i \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t = \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t e_t^i$$
$$c_t^i \sum_{t=0}^{\infty} \beta^t = \sum_{t=0}^{\infty} \beta^t e_t^i$$
$$c_t^i = (1-\beta)\sum_{t=0}^{\infty} \beta^t e_t^i \tag{7}$$

Equation (7) is a famous result in macro credited to Milton Friedman that Friedman called the "Permanent Income Hypothesis." In this model, households consume a constant share of their total lifetime wealth, known as "permanent income."

Now we can replace for each type's endowment sequence to solve for allocations. For $i = 1$, $e_t^1 = (2, 0, 2, 0, \dots)$, so

$$\begin{aligned}\sum_{t=0}^{\infty} \beta^t e_t^1 &= 2 + 0 + 2\beta^2 + 0 + 2\beta^4 + \dots \\ &= \sum_{t=0}^{\infty} 2\beta^{2t} \\ &= 2\sum_{t=0}^{\infty} \left(\beta^2\right)^t\end{aligned}$$

Hence

$$\begin{aligned}c_t^1 &= 2(1-\beta)\sum_{t=0}^{\infty} \left(\beta^2\right)^t \\ c_t^1 &= \frac{2(1-\beta)}{1-\beta^2} \\ c_t^1 &= \frac{2}{1+\beta}\end{aligned} \tag{8}$$

where the final line follows by $1 - \beta^2 = (1-\beta)(1+\beta)$.

Similarly, for $i = 2$, $e_t^2 = (0, 2, 0, 2, \dots)$, so

$$\begin{aligned}\sum_{t=0}^{\infty} \beta^t e_t^2 &= 0 + 2\beta + 0 + 2\beta^3 + \dots \\ &= 2\beta\sum_{t=0}^{\infty} \left(\beta^2\right)^t\end{aligned}$$

Hence

$$\begin{aligned}c_t^2 &= 2\beta(1-\beta)\sum_{t=0}^{\infty} \left(\beta^2\right)^t \\ c_t^2 &= \frac{2\beta}{1+\beta}\end{aligned} \tag{9}$$

We can use the intertemporal budget constraints for type i households to solve for equilibrium savings:

$$a_{t+1}^i = (1+r)a_t^i + e_t^i - c_t^i$$

Then for $i = 1$,

$$t = 0: \qquad a_1^1 = 2 - \frac{2}{1+\beta} = \frac{2\beta}{1+\beta}$$

$$t = 1: \qquad a_2^1 = (1+r) \cdot \frac{2\beta}{1+\beta} - \frac{2}{1+\beta} = \frac{1}{\beta} \cdot \frac{2\beta}{1+\beta} - \frac{2}{1+\beta} = 0$$

$$t = 2: \qquad a_3^1 = 2 - \frac{2}{1+\beta} = \frac{2\beta}{1+\beta}$$

$$t = 3: \qquad a_4^1 = (1+r) \cdot \frac{2\beta}{1+\beta} - \frac{2}{1+\beta} = \frac{1}{\beta} \cdot \frac{2\beta}{1+\beta} - \frac{2}{1+\beta} = 0$$

And so on. Noting the pattern, we see that in equilibrium,

$$a_{t+1}^1 = \begin{cases} \frac{2\beta}{1+\beta} & \text{for } t = 0, 2, 4, \dots \\ 0 & \text{for } t = 1, 3, 5, \dots \end{cases} \tag{10}$$

By the market clearing condition in the securities market,

$$a_{t+1}^2 = \begin{cases} \frac{-2\beta}{1+\beta} & \text{for } t = 0, 2, 4, \dots \\ 0 & \text{for } t = 1, 3, 5, \dots \end{cases} \tag{11}$$

Equation (6) gives the equilibrium gross return. Equations (8), (9), (10), (11) give the equilibrium allocations.

Just like in our two-period economy, in initial period, the poor type borrows their consumption from the rich type. In the following period, when the rich type has no endowment, he consumes the loan repayment from the poor type. This two-period cycle repeats forever and ever in the infinite horizon.

Exercise. Verify that equilibrium rate of return and allocations satisfy all budget constraints and market clearing conditions.

Exercise. Verify that equilibrium savings satisfies the condition assumed earlier: $\lim_{T\to\infty} \frac{a_{T+1}}{(1+r)^T} = 0$.

Exercise. Solve for allocation, savings/borrowing in an economy with CRRA preferences and endowment growth. That is, let

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}; \qquad e_t^1 = \begin{cases} 2\gamma^t & \text{for } t = 0, 2, 4, \dots \\ 0 & \text{for } t = 1, 3, 5, \dots \end{cases}; \qquad e_t^2 = \begin{cases} 0 & \text{for } t = 0, 2, 4, \dots \\ 2\gamma^t & \text{for } t = 1, 3, 5, \dots \end{cases}$$

Note that the competitive equilibrium we solve above is a special case where $\sigma = 1$ and $\gamma = 1$. Note that when there is endowment growth, we need to write the Ponzi scheme constraint as $a_{t+1} \geq -\bar{A}\gamma^t$ for all t, some constant $\bar{A} > 0$, to allow for the amount that can be borrowed to grow at the same rate as endowments.