ECON 8040 – Midterm Exam Prep

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Midterm Exam

- Lecture Notes
  - Competitive Equilibrium & Welfare Theorems
  - Economic Growth (Solow Model)
- Problem Sets 1, 2 & 3

Competitive Equilibrium

- Carefully define all equilibrium objects
  - Allocations and prices
  - Clarify what agents take as given, what are their choices, and what their objectives are
- Solve for all equilibrium objects!
  - Don't forget prices when using Neghishi Method!
  - Solve for all states in stochastic economy.
- Keep tidy notation to track states in stochastic economy
  - e.g., $c_t^1(s^t)$, not c_t^1
  - e.g., $p_t(s_1)$, not p_t

Welfare Theorems

- First Welfare Theorem
  - Pareto Efficiency (PE)
- Second Welfare Theorem
  - Definition of transfers (only occurs once, not every period)
  - Planner's Problem (PP)
  - Negishi Method for solving CE
- Don't confuse definitions of PE and PP!

Growth Models

- ⋆ Production functions
- ⋆ Law of motion of capital
- ⋆ Steady state and its determinants

Proving "If and Only If"

- "if and only if" requires two proofs!
- $P \Leftrightarrow Q$
  1. $P \Rightarrow Q$
     - i. Assume P
     - ii. Need to show (NTS) Q holds
  2. $Q \Rightarrow P$
     - i. Assume Q
     - ii. NTS P holds

Proving "If and Only If"

- E.g., ADCE $\Leftrightarrow$ SMCE
  1) SM $\Rightarrow$ AD
     i. Assume SMCE $\{c_t^1, c_t^2, a_{t+1}^1, a_{t+1}^2\}_{t=0}^{\infty}$, $\{r_{t+1}\}_{t=0}^{\infty}$, and $-\bar{A}$
     ii. NTS there exists price sequence $\{p_t\}_{t=0}^{\infty}$ s.t. HH makes same allocation in ADCE.
     iii. Use interest rate r_{t+1} to define AD prices p_t
     iv. Use period budget constraints, and borrowing limit to write present value budget constraint
     v. Conclude HH in AD faces same problem as in SM, so make same choices.
  2) AD $\Rightarrow$ SM
     i. Assume ADCE $\{c_t^1, c_t^2, p_t\}_{t=0}^{\infty}$
     ii. NTS $\exists$ interest rate r_{t+1}, asset holding a_{t+1}^i, borrowing limit $-\bar{A}$
     iii. Define interest rates in terms of AD prices p_t
     iv. Write equation for a_{t+1} in terms of prices, consumption, endowments
     v. Show $\exists$ $\bar{A} > 0$ s.t. $a_{t+1} > -\bar{A}$ for all t

Proof by Contradiction

- Logically, $P \Rightarrow Q$ is negated by $P \wedge \neg Q$
- Proof by contradiction shows $P \Rightarrow Q$ by negating $P \wedge \neg Q$
  - Do not try to show $\neg P \wedge Q$!
  - $P \Rightarrow Q$, yet Q could happen in absence of P
- E.g., show CE allocation is PE (i.e., CE $\Rightarrow$ PE)
- Easier to negate statement "Allocation is $CE \wedge \neg PE$"
- Sketch
  i. Assume allocation is CE and not PE (i.e., $CE \wedge \neg PE$)
  ii. Use definition of PE to show alternative allocation that is Pareto improving cannot exist
  iii. Conclude $CE \wedge \neg PE$ does not hold, thus CE $\Rightarrow$ PE
  - See sketch of proof of Proposition 2 on page 7 of "Introduction to Competitive Equilibria and Welfare Theorems" lecture notes

Problem 3

Prove properties of CRRA utility

$$U(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

a) Let $f(\sigma) = c^{1-\sigma} - 1$, $g(\sigma) = 1 - \sigma$. Both are differentiable and $\lim_{\sigma \to 1} \frac{f(\sigma)}{g(\sigma)} = \frac{0}{0}$. By l'Hôspital's rule,

$$\lim_{\sigma \to 1} U(c) = \lim_{\sigma \to 1} \frac{f'(\sigma)}{g'(\sigma)} = \log c$$

b) Plug derivatives into definition to show

$$-\frac{U''(c)}{U'(c)} c = \sigma$$

Problem 3

c) Plug into definition

$$\text{IES} \equiv -\frac{\%\Delta c}{\%\Delta U'(c)} = -\frac{dc}{c} \Bigg/ \frac{dU'(c)}{U'(c)} = -\frac{U'(c)}{c\left(\frac{dU'(c)}{dc}\right)} = -\frac{U'(c)}{c \times U''(c)}$$

$\rightarrowtail$ Sensitivity of intertemporal substitution wrt 1% change in MRS

d) Use $U'(c)$, $U''(c)$ to show Inada conditions

i. strictly increasing
ii. strictly concave
iii. $\lim_{c\to 0} U'(c) = +\infty$
iv. $\lim_{c\to +\infty} U'(c) = 0$

Problem 3

e) Show MRS is invariant to scaling of consumption.

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

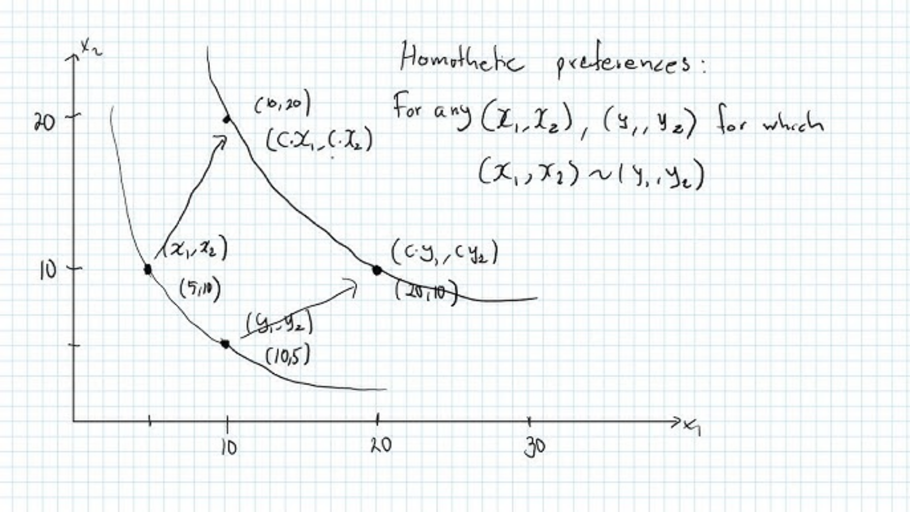
subject to $\sum_{t=0}^{\infty} p_t c_t \leq \sum_{t=0}^{\infty} p_t e_t; \gamma$
FOCs wrt c_{t+s}, c_t

- $\beta^{t+s} c_{t+s}^{-\sigma} = \gamma p_{t+s}$
- $\beta^t c_t^{-\sigma} = \gamma p_t$

For $\lambda > 0$,

$$MRS(c_{t+s}, c_t) = \beta^s \left(\frac{c_{t+s}}{c_t}\right)^{-\sigma} = MRS(\lambda c_{t+s}, \lambda c_t)$$

Problem 3, part e



Problem 3

f) Given $\{\bar{c}_t\}_{t=0}^{\infty}$ is CE allocation when income equals y. Need to show allocation $\{\widetilde{c}_t\}_{t=0}^{\infty} = \{\lambda \bar{c}_t\}_{t=0}^{\infty}$ when income $\widetilde{y} = \lambda \bar{y}$

- Affordable using the budget constraint
- Optimal using homotheticity of CRRA utility (i.e., MRS invariant to scaling of consumption)

Lecture Note Questions – Problem 4

1. Exercises with following Arrow-Debreu economy:
ADCE is allocation $\{c_t^1, c_t^2\}_{t=0}^{\infty}$ and prices $\{p_t\}_{t=0}^{\infty}$ such that
   A. Given prices, household $i = 1, 2$ makes allocation that solves

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log c_t^i$$

   s.t. $\sum_{t=0}^{\infty} p_t c_t^i = \sum_{t=0}^{\infty} p_t e_t^i$

   B. For $t = 0, 1, 2, \ldots,$ market clears

$$c_t^1 + c_t^2 = e_t^1 + e_t^2$$

where $e_t^1 = (2, 0, 2, 0, \ldots)$ and $e_t^2 = (0, 2, 0, 2, \ldots)$

Lecture Note Questions

a) Show prices are *not* constant. E.g., prove by contradiction

- Suppose $p_t = p$ for $t = 0, 1, 2, \ldots$

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t^i)$$

  s.t. $\sum_{t=0}^{\infty} pc_t^i = \sum_{t=0}^{\infty} pe_t^i; \lambda$

- By FOCs, $\lambda = \frac{\beta^t}{c_t^i} = \frac{\beta^{t+1}}{c_{t+1}^i} \Leftrightarrow c_t^i = \beta c_{t-1}^i = \beta^t c_0^i$
- By budget constraint,
  $\sum_{t=0}^{\infty} \beta^t pc_0^i = \frac{pc_0^i}{(1-\beta)} = \sum_{t=0}^{\infty} pe_t^i \Leftrightarrow c_0^i = (1-\beta) \sum_{t=0}^{\infty} e_t^i$
- Contradiction: HH cannot consume infinite resources in first period

Lecture Note Questions

b) Solve ADCE

- Derive Euler equation using household i's first-order conditions

$$c_{t+1}^i = \beta \frac{p_t}{p_{t+1}} c_t^i$$

- Write c_t^i in terms of c_0^i using Euler

$$c_t^i = \beta \frac{p_{t-1}}{p_t} c_{t-1}^i = \beta \frac{p_{t-1}}{p_t} \times \beta \frac{p_{t-2}}{p_{t-1}} c_{t-2}^i = \cdots = \beta^t \frac{p_0}{p_t} c_0^i$$

- Normalize $p_0 = 1$. Write c_0^i in terms of discount factor, prices, and endowments using budget constraint

$$\sum_{t=0}^{\infty} p_t c_t^i = \sum_{t=0}^{\infty} p_t \beta^t \frac{p_0}{p_t} c_0^i = c_0^i \sum_{t=0}^{\infty} \beta^t = \sum_{t=0}^{\infty} p_t e_t^i$$

$$\Rightarrow c_t^i = (1-\beta) \frac{\beta^t}{p_t} \sum_{t=0}^{\infty} p_t e_t^i$$

Lecture Note Questions

b) Solve ADCE

- Solve p_t using the market-clearing condition

$$c_t^1 + c_t^2 = (1-\beta)\frac{\beta^t}{p_t}\sum_{t=0}^{\infty} p_t(e_t^1 + e_t^2)$$

$$2 = 2(1-\beta)\frac{\beta^t}{p_t}\sum_{t=0}^{\infty} p_t$$

$$p_t = \beta^t$$

- Solve allocations $\hat{c}_t^1$, $\hat{c}_t^2$ by plugging p_t, endowments in expression for c_t^i (be careful with sums!)

$$\hat{c}_t^i = (1-\beta)\sum_{t=0}^{\infty} \beta^t e_t^i$$

Lecture Note Questions

2. Prove equivalence of ADCE and SMCE
   Exercise (optional): Show $1 + r_{t+1} = \frac{1}{\beta}$, $c_t^1 = \frac{2}{1-\beta}$, $c_t^2 = \frac{2\beta}{1-\beta}$ in SMCE with same preferences, endowments as ADCE in Definition 3.

   SMCE is allocation $\{c_t^1, a_{t+1}^1, c_t^2, a_{t+1}^2\}_{t=0}^{\infty}$ and interest rate $\{r_{t+1}\}_{t=0}^{\infty}$ such that

   A. Given prices, household $i = 1, 2$ makes allocation that solves

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log c_t^i$$

   s.t. $c_t^i + a_{t+1}^i = e_t^i + (1 + r_t)a_t^i$

   B. For $t = 0, 1, 2, \ldots,$ markets clear
      i. Consumption good: $c_t^1 + c_t^2 = e_t^1 + e_t^2$
      ii. Securities: $a_{t+1}^1 + a_{t+1}^2 = 0$

Lecture Note Questions

3. Prove First Welfare Theorem: Let $\{c_t^1, c_t^2\}_{t=0}^{\infty}$ be CE. Then it is Pareto efficient.
   Suppose not. If CE allocation is not PE, then $\exists$ allocation $\{\hat{c}_t^1, \hat{c}_t^2\}_{t=0}^{\infty}$ such that
   i. $\hat{c}_t^1 + \hat{c}_t^2 = 2$ for $t = 0, 1, 2, \dots$
   ii. $\sum_{t=0}^{\infty} \beta^t \log(\hat{c}_t^1) > \sum_{t=0}^{\infty} \beta^t \log(c_t^1)$
   iii. $\sum_{t=0}^{\infty} \beta^t \log(\hat{c}_t^2) = \sum_{t=0}^{\infty} \beta^t \log(c_t^2)$

   See sketch of proof of Proposition 2 on page 7 of "Introduction to Competitive Equilibria and Welfare Theorems" lecture notes

Lecture Note Questions

4. Allocation is PE **if and only if** that allocation solves PP
   - Variation in scores explained by setting up proof correctly
   - Show $PE \Rightarrow PP$
     - Assume PE allocation $\{c_t^1, c_t^2\}_{t=0}^{\infty}$
     - NTS $\exists\ \alpha$ s.t. solution to PP is $\{c_t^1, c_t^2\}_{t=0}^{\infty}$
   - Show $PP \Rightarrow PE$
     - Assume Planner chooses $\{c_t^1, c_t^2\}_{t=0}^{\infty}$ for some weight α
     - Suppose Planner allocation is not PE.
     - NTS $PP \wedge \neg PE$ is not true. Thus $PP \Rightarrow PE$

Lecture Note Questions

5. Find α corresponding to ADCE in lecture notes.
   - Use model economy from Question 1 in lecture notes
   - Write down Planner's allocations $c_t^1(\alpha), c_t^2(\alpha)$.
   - Find α such that $c_t^1(\alpha) = (c_t^1)_{CE}$, $c_t^2(\alpha) = (c_t^2)_{CE}$
6. Negishi Method
   a) Find allocations as function of α (like you did in Q5). Write Lagrange multiplier $\pi_t(\alpha)$, too
   b) Write transfer functions using $\pi_t(\alpha)$, $c_t^i(\alpha)$, e_t^i
   c) Show $t^i(\alpha)$ is HD1 and sums to zero
      - Only ratio of weights affects allocation
      - Transfers must sum to zero for feasibility to hold
   d) Find α_{CE} such that $t^1(\alpha_{CE}) = t^2(\alpha_{CE}) = 0$.
      - Equilibrium allocations $c_t^i = c_t^i(\alpha_{CE})$
      - Equilibrium prices $p_t = \pi_t(\alpha_{CE})$

Lecture Note Questions

7. Stochastic ADCE
   a) Prove First Welfare Theorem
      - Show by contradiction as before
      - Be careful with s^t notation, but logic of proof is same
   b) Solve ADCE when $\pi(1) = \pi(2) = 1/2$
      - Carefully define ADCE for two-state economy
      - Solving much the same: use Euler, budget constraint, MCC, and MRS
      - Write **all** equilibrium allocations and prices
8. Asset Prices
   - Write prices like $q_t(s^t, \eta_j)$, $q_{t+1}((s^t, \eta_j = 2), \eta_j = 1)$ to answer question
   - Think of how many assets (and which ones) need to be bought in period t and $t + 1$
9. See notes above about showing SMCE, ADCE equivalence

Problem 2

a) Define ADCE
- Each consumer $i = 1, 2$ maximizes *expected utility* s.t. 1 budget constraint
- 1 time period; 3 possible state of endowments
- Thus 3 MCCs

b) Define Pareto efficient allocation
- Feasible
- Cannot increase one agent's utility without decreasing another's utility
- **Do not define Planner's Problem!**

c) Show ADCE allocation is PE
- How should we prove this?

Problem 2

d) Solve Stochastic ADCE

- WLOG normalize $p(s_1) = 1$
- FOCs wrt 3 choice variables for household i
- Replace prices on LHS of budget constraint
- Write $c^i(s_1)$ in terms of prices, endowments
- Use FOCs to write $c^i(s_2), c^i(s_3)$ in terms of prices, endowments
- Plug $c^i(s_j)$ for $i = 1, 2$, $j = 1, 2, 3$ into MCCs to solve prices
    - You'll have 3 equations for 2 unknowns, but solution is unique
- Solve allocations by replacing prices into equations for $c^i(s_j)$ you derived earlier

Question 1

a) Suppose there are j firms in the economy that each hold α_j share of aggregate capital K and aggregate labor L. Use CRS property to show $\sum_j F(K_j, L_j) = F(K, L)$

b)

$$\text{Capital Share} := \frac{rK}{Y}$$

$$\text{Labor Share} := \frac{wL}{Y}$$

Question 1

c) $\pi = F(K, L) - wL - rK$

- Hint: Start with Cobb-Douglas p.f. to get the answer. Then prove it more generally **for any CRS p.f.**
- Sketch
  - Take total derivative of $F(K, L)$
  - Fix dK, dL to correspond to doubling of K, L
  - Use CRS property to conclude what this implies about dF
  - Replace partial derivatives with factor prices
  - Calculate profit

Questions 2–4

- Exercises with Penn World Tables data