ECON 8040 – TA15

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Today's Session

- Final exam **Thursday, December 7, 3:30–6:30 p.m.**
- PS8, PS9, CE Grades

Problem 1

(a) Define SMCE in "Lucas Tree" economy

A. HH maximizes utility subject to
  - budget constraint
  - no Ponzi schemes

B. Markets clear
  i. Consumption good: $c_t = d_t$ for $t = 0, 1, \ldots$
  ii. Shares: $s_t = 1$ for $t = 0, 1, \ldots$
  iii. Assets: $b_t = 0$ for $t = 0, 1, \ldots$

Problem 1

(b) Recursive Competitive Equilibrium is collection of price functions $\{p^s(d), p^b(d)\}$, value function $\{V^*(w,d)\}$, and household policy functions $\{c^*(w,d), b^*(w,d), s^*(w,d)\}$ such that

A. Given price functions and dividends d and d', household policy functions solve its Bellman equation

$$V(w,d) = \max_{c,b',s'} \{u(c) + \beta V(w',d')\}$$

subject to

$$\begin{aligned} c + p^b(d)b' + p^s(d)s' &\leq w, \\ c, s' &\geq 0 \\ b' &\geq -\bar{A} \text{ for some } \bar{A} > 0 \end{aligned}$$

The solution to the Bellman equation is value function $V^*(w,d)$.

Problem 1

B. Price functions are functions of dividends d.

$$p^s = p^s(d)$$
$$p^b = p^b(d)$$

C. Allocation is feasible. That is, household policies satisfy market clearing conditions in aggregate. For all dividends d,

$$c(d, d) = d$$
$$b(d, d) = 0$$
$$s(d, d) = 1$$

Problem 1

(c) Assume $d_t = 1$ and find p_t^b

- HH FOCs wrt b_{t+1}, c_t, and c_{t+1}

$$p_t^b = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

- By MCC, $c_t = c_{t+1} = 1$, so

$$p_t^b = \beta$$

Problem 1

(c) Assume $d_t = 1$ and find p_t^s

- HH FOCs wrt k_{t+1}, c_t, and c_{t+1}

$$p_t^s = \beta \frac{u'(c_{t+1})}{u'(c_t)} \left(p_{t+1}^s + d_{t+1}\right)$$

- Iterate forward r periods

$$p_t^s = \beta^r p_{t+r}^s + \sum_{k=1}^{r} \beta^k$$

- Assume no speculative bubble: $\lim_{r\to\infty} \beta^r p_{t+r}^s = 0$

$$p_t^s = \sum_{k=1}^{\infty} \beta^k$$

Problem 1

(d) Assume $u(c_t) = \log(c_t)$ and $d_t = \begin{cases} 1 & \text{if } t = 0, 2, 4, \ldots \\ 2 & \text{if } t = 1, 3, 5, \ldots \end{cases}$

- Solve for p_t^b, p_t^s same as (c)
- Two prices for bonds, stocks (odd/even periods)

Problem 2

(a) Define SMCE

- Prices, allocations, and *policy* $\{\tau_c, \tau_k, \tau_n, T_t\}$
- HH budget constraint
  - Consumption tax *adds* to expenditure
  - Labor, capital taxes *subtract* from income
- Firm maximizes profit
- Government *balances budget*
  - Issues no debt B_t
  - Makes no expenditures g_t
- Four markets clear
  - Consumption good: $c_t + x_t = y_t$
  - Asset market also clears

Problem 2

$$u(c_t, 1-n_t) = \frac{\left(c_t^{\mu}(1-n_t)^{1-\mu}\right)^{1-\sigma}}{1-\sigma}$$

$$\begin{aligned} u_c(c_t, 1-n_t) &= \mu \left[c_t^{\mu}(1-n_t)^{1-\mu}\right]^{-\sigma} c_t^{\mu-1}(1-n_t)^{1-\mu} \\ &= \frac{\mu \left[c_t^{\mu}(1-n_t)^{1-\mu}\right]^{1-\sigma}}{c_t} \\ u_n(c_t, 1-n_t) &= -(1-\mu)\left[c_t^{\mu}(1-n_t)^{1-\mu}\right]^{-\sigma} c_t^{\mu}(1-n_t)^{-\mu} \\ &= \frac{-(1-\mu)\left[c_t^{\mu}(1-n_t)^{1-\mu}\right]^{1-\sigma}}{1-n_t} \end{aligned}$$

Problem 2

(b) Write i_{t+1} in terms of taxes and allocations

- FOC wrt a_{t+1}:

$$1 + i_{t+1} = \frac{\lambda_t}{\lambda_{t+1}}$$

- **Option #1:** Replace λ_t w/ FOC wrt c_t, λ_{t+1} w/ FOC wrt c_{t+1}
- **Option #2:** Replace $\frac{\lambda_t}{\lambda_{t+1}}$ using FOC wrt k_{t+1}

Problem 2

(c) Write 3 equations characterizing **steady state**

- Impose $c_t = c_{t+1}$, $k_t = k_{t+1}$, so on!
- Euler equation uses FOCs wrt k_{t+1}, c_t, c_{t+1}
- Aggregate feasibility $\neq$ HH budget constraint
- MRS uses FOCs wrt n_t, c_t

(d) Use Euler from (c) to solve $\frac{k}{n}$ in terms of parameters, taxes **only**

(e) Use equation from (b) to solve i^*

- **Option #1:** Impose $c_t = c_{t+1}$, $n_t = n_{t+1}$
- **Option #2:** Plug in $\frac{k}{n}$ from (d)

Problem 1

Optimal capital taxation / subsidy

(a) ADCE is prices $\{p_t, i_t\}_{t=0}^{\infty}$, policy $\{\tau_c, \tau_k, g, G_t\}_{t=0}^{\infty}$, and allocations $\{c_t, x_t, k_{t+1}, y_t, k_t^d\}_{t=0}^{\infty}$ such that

A. Given prices and policy, allocations $\{c_t, x_t, k_{t+1}\}_{t=0}^{\infty}$ solve

$$\max_{\{c_t, x_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

subject to

$$\sum_{t=0}^{\infty} p_t[(1+\tau_c)c_t + x_t] = \sum_{t=0}^{\infty} p_t i_t (1-\tau_k) k_t$$

$$k_{t+1} = x_t + k_t \text{ for all } t$$

$$c_t, x_t, k_{t+1} \geq 0 \text{ for all } t$$

$$k_0 \text{ given}$$

Problem 1

(a) Define ADCE (cntd.)

B. Given prices and policy, firm allocations $\{y_t, k_t^d\}_{t=0}^{\infty}$ maximize profit:

$$\max_{\{y_t, k_t^d\}_{t=0}^{\infty}} y_t - i_t k_t^d$$

subject to $y_t = rk_t$ for all t

C. Government budget balances. For all t,

$$\tau_c c_t + \tau_k k_t = G_t = g y_t$$

D. Markets clear. For all t,

  i. Consumption good: $c_t + x_t + G_t = y_t$
  ii. Capital: $k_t^d = k_t$

Problem 1

(b) Derive Euler equation

$$\frac{1}{c_t} = \frac{\beta[1 + r(1 - \tau_k)]}{c_{t+1}}$$

(c) Find long-run growth rate of economy

$$\frac{c_{t+1}}{c_t} = \beta[1 + r(1 - \tau_k)]$$

(d) Can government increase growth by subsidizing capital? Yes.

(e) Optimal capital subsidy is $\tau_k = 0$

Problem 2

Infinite-horizon production economy with elastic labor supply

(a) Write aggregate feasibility
In aggregate variables,

$$\bar{N}c_t + K_{t+1} = K_t^{\alpha} H_t^{1-\alpha} + (1-\delta)K_t$$

In per-person variables,

$$c_t + k_{t+1} = k_t^{\alpha} h_t^{1-\alpha} + (1-\delta)k_t$$

Problem 2

(b) Write profit-maximization problem and derive equations for r_t, w_t.
Taking prices r_t and w_t as given, firm chooses K_t, H_t such that

$$\max_{\{K_t, H_t\}_{t=0}^{\infty}} K_t^{\alpha} H_t^{1-\alpha} - r_t K_t - w_t H_t$$

Firm's optimality conditions are

$$r_t = \alpha \left(\frac{K_t}{H_t} \right)^{\alpha-1} = \alpha \left(\frac{\bar{N} k_t}{\bar{N} h_t} \right)^{\alpha-1} = \alpha \left(\frac{k_t}{h_t} \right)^{\alpha-1}$$

$$w_t = (1-\alpha) \left(\frac{k_t}{h_t} \right)^{\alpha}$$

Problem 2

(c) Define SMCE.

- $\rightarrowtail$ Government budget: $\bar{N}Tr_t = \tau w_t H_t$
- $\rightarrowtail$ Consumption MCC: $c_t + k_{t+1} = k_t^{\alpha} h_t^{1-\alpha} + (1-\delta)k_t$
    - Government only makes *transfers*, distinct from *expenditure*, so government does not enter aggregate feasibility

(d) 3 eqns (in per-person terms) that describe equilibrium allocations

$$\text{Feasibility:} \quad c_t + k_{t+1} = k_t^{\alpha} h_t^{1-\alpha} + (1-\delta)k_t$$

$$\text{MRS:} \quad \frac{(1-\phi)c_t}{\phi(1-h_t)} = (1-\tau)(1-\alpha)\left(\frac{k_t}{h_t}\right)^{\alpha}$$

$$\text{Euler:} \quad \frac{c_{t+1}}{c_t} = \beta\left[\alpha\left(\frac{k_t}{h_t}\right)^{\alpha-1} + 1 - \delta\right]$$

Problem 2

(e) Find steady-state $\frac{K_t}{H_t}$. How is it affected by τ?

Use Euler equation and impose steady state condition:

$$\frac{k}{h} = \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{1}{1-\alpha}}$$

Tax rate τ does *not* affect long-run ratio of per-person capital to per-person hours worked.

Problem 2

(f) How does the tax rate τ affect wage, rental rate, capital-to-output ratio, hours worked and output in the economy?

- Wage, rental rate, and capital-to-output ratio are *unaffected* by τ
- Hours worked and output are *decreasing* in τ

Problem 3

AKH endogoneous growth model with human capital accumulation

(a) Tax-distorted competitive equilibrium is prices $\{w_t, r_t\}_{t=0}^{\infty}$, household allocation $\{c_t, x_t^k, x_t^h, k_{t+1}, h_{t+1}\}_{t=0}^{\infty}$, firm allocation $\{y_t, k_t^d, h_t^d\}_{t=0}^{\infty}$, and policy $\{g_t, \tau_n, \tau_k\}_{t=0}^{\infty}$ such that

A. Given prices and policy, household allocations solves preference maximization:

$$\max_{\{c_t, x_t^k, x_t^h, k_{t+1}, h_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

subject to

$$c_t + x_t^k + x_t^h = (1-\tau_n) w_t h_t + (1-\tau_k) r_t k_t$$
$$k_{t+1} = x_t^k + (1-\delta_k) k_t$$
$$h_{t+1} = x_t^h + (1-\delta_h) h_t$$

Problem 3

B. Given prices and policy, firm allocation maximizes profit:

$$\max_{\{y_t, k_t^d, h_t^d\}_{t=0}^{\infty}} y_t - r_t k_t^d - w_t h_t^d$$

subject to $y_t = (k_t^d)^{\alpha}(h_t^d)^{1-\alpha}$

C. Government budget balances $\tau_n w_t h_t + \tau_k r_t k_t = g_t$

D. Markets clear
   i. Consumption: $c_t + x_t^k + x_t^h + g_t = y_t$
   ii. Physical capital: $k_t = k_t^d$
   iii. Human capital: $h_t = h_t^d$

Problem 3

(b) Find growth rate

- ⋆ Derive two inter-termporal optimality conditions (for each form capital) and assume $\delta_k = \delta_h$ to solve:

$$\frac{c_{t+1}}{c_t} = \beta \left\{ [(1-\tau_k)\alpha]^{\alpha} [(1-\tau_n)(1-\alpha)]^{1-\alpha} + 1 - \delta \right\}$$

Problem 3

(c) Define TDCE with tax rebate on human capital investment x_t^h. Similar to (a), except:

- Household's budget constraint:

$$c_t + x_t^k + x_t^h = (1 - \tau_n) w_t h_t + (1 - \tau_k) r_t k_t + \tau_n x_t^h$$

- Government budget condition:

$$\tau_n w_t h_t + \tau_k r_t k_t - \tau_n x_t^h = g_t$$

(d) Growth rate with tax rebate in place:

$$\frac{c_{t+1}}{c_t} = \beta \left\{ (1 - \alpha)^{1-\alpha} [(1 - \tau_k)\alpha]^{\alpha} + 1 - \delta \right\}$$

(e) The economy in part (c), with tax rebate on human capital investment, grows faster.

Problem 4

Endogoneous fertility model

(a) Write aggregate feasibility in terms of aggregate variables

$$C_t + K_{t+1} + N_{t+1}(\theta + wb) = wN_t + (R+1)K_t$$

(b) Write planning problem of head of dynasty that maximizes generation zero's welfare.

$$\max_{\{C_t, K_{t+1}, N_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t N_t^{\eta+\sigma} \frac{C_t^{1-\sigma}}{1-\sigma}$$

subject to

$$C_t + K_{t+1} + N_{t+1}(\theta + wb) = wN_t + (R+1)K_t \text{ for } t = 0, 1, \ldots$$
$$K_0 \text{ given}, N_0 \text{ given}$$

Problem 4

(c) Derive FOCs. Solve for per-capita variables c_t and n_t. How do they depend on initial capital and population?
The FOCs of planner are:

$$\begin{aligned}
\text{wrt } C_t: &\quad \lambda_t = \beta^t N_t^{\eta+\sigma} C_t^{-\sigma} \\
\text{wrt } K_{t+1}: &\quad \lambda_t = \lambda_{t+1}(R+1) \\
\text{wrt } N_{t+1}: &\quad \lambda_t(\theta + wb) = \lambda_{t+1} w + \beta^{t+1}(\eta+\sigma) N_{t+1}^{\eta+\sigma-1} \frac{C_{t+1}^{1-\sigma}}{1-\sigma}
\end{aligned}$$

Problem 4

(c) (cntd.) Using the Euler equations that describe intertemporal optimality of K_{t+1} and N_{t+1}, we solve

$$c_t = \frac{(R+1)(\theta + wb)(1-\sigma) - w(1-\sigma)}{\eta + \sigma} \quad n_t = \left(\frac{1}{\beta(R+1)}\right)^{\frac{1}{\eta}}$$

Per-capita consumption and fertility *do not depend* on initial capital stock or initial population.

Problem 4

(d) Write planner's problem as Bellman. Prove that it is a contraction mapping.

$$V(K,N) = \max_{K',N'} \left\{ N^{\eta+\sigma} \frac{(wN + (R+1)K - K' - N'(\theta + wb))^{1-\sigma}}{1-\sigma} + \beta V(K',N') \right\}$$

⟶ It can be shown this Bellman satisfies Blackwell's sufficient conditions for a contraction mapping

Problem 4

(e) Show $V(K,N) = N^{1+\eta}v(k)$. Write down Bellman that $v(k)$ solves: Replace for $V(K,N)$ in Bellman from (d) and do algebra to write everything in per-capita terms:

$$v(k) = \max_{k',n} \left\{ \frac{(w + (R+1)k - k'n - n(\theta + wb))^{1-\sigma}}{1-\sigma} + \beta n^{1+\eta} v(k') \right\}$$

Problem 4

(f) Write FOCs for k' and n. Show k' is constant and independent of k. The FOCs are

$$\text{wrt } n: \quad [w + (R+1)k - k'n - n(\theta + wb)]^{-\sigma}(k' + \theta + wb) = \beta(1+\eta)n^{\eta}v(k')$$
$$\text{wrt } k': \quad n[w + (R+1)k - k'n - n(\theta + wb)]^{-\sigma} = \beta n^{1+\eta}v'(k')$$

Using feasibility,

$$k^* = \frac{c^* + n^*(\theta + wb) - w}{R + 1 - n^*}$$

where c^*, n^* are optimal per-capita consumption and fertility from (c). Both are constant, and neither depends on state k, so k' is *constant and independent of* k, too.

Computation Exercise

(a) Write stationary planner's problem
- Define $c_t \equiv \frac{C_t}{(1+\gamma)^t N_t}$ and $k_t \equiv \frac{K_t}{(1+\gamma)^t N_t}$

(b) Write problem recursively

$$V(k_t) = \max_{c_t, k_{t+1} \geq 0} \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} + (1+\gamma)^{1-\sigma} \beta V(k_{t+1}) \right\}$$

subject to

$$c_t + (1+\gamma)(1+\eta)k_{t+1} = Ak_t^{\alpha} + (1-\delta)k_t$$
$$c_t, k_{t+1} \geq 0, k_0 \text{ given}$$

Solve for k_{ss}
- Derive Euler equation from FOCs
- c_t, k_t constant in steady state

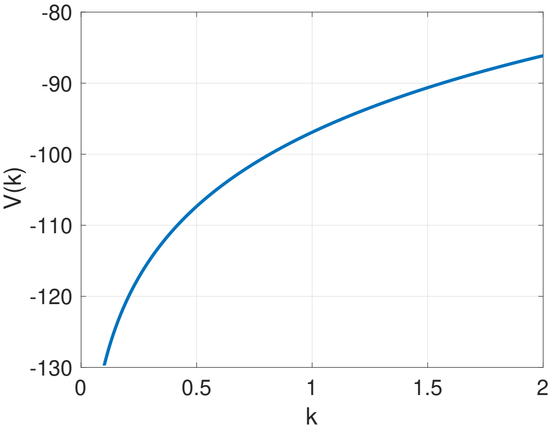
Computation Exercise

(c) Calibrate parameters of the model. Given σ, η, and γ,

1. $\alpha = \frac{r_t K_t}{Y_t} = 1/3$
2. Use $\dfrac{K_{t+1} - (1-\delta)K_t}{Y_t} = 0.21$ and $\dfrac{K_t}{Y_t} = 3$ to solve for δ
3. Normalize $k^* = 1$. Use $\alpha \frac{Y_t}{K_t}$ to solve for A
4. Use Euler equation to solve β

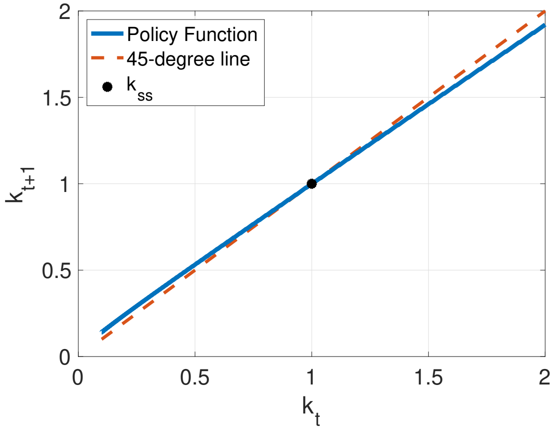
Computation Exercise

(d) Plot value function $V(k)$



Computation Exercise

(d) Plot policy function $g(k)$

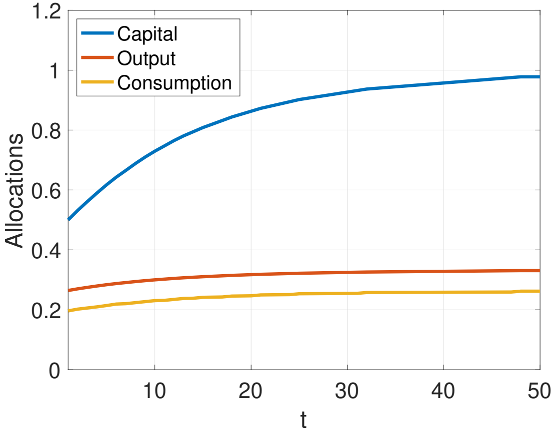


Computation Exercise

(e) Simulate 50 periods where $k_0 = 0.5k_{ss}$
- $\rightarrowtail$ Use $g(k)$ to generate capital $\{k_1, \ldots, k_{50}\}$
- $\rightarrowtail$ $y_t = Ak_t^{\alpha}$ gives $\{y_1, \ldots, y_{50}\}$
- $\rightarrowtail$ Feasibility gives $\{c_1, \ldots, c_{50}\}$

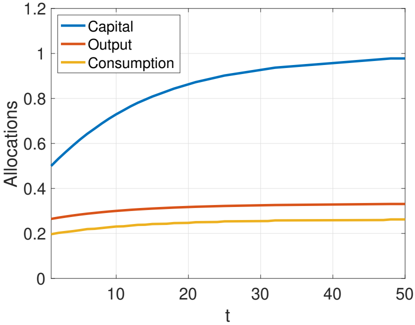
Computation Exercise

(e) Simulate 50 periods where $k_0 = 0.5k_{ss}$



Computation Exercise

(e) Simulate 50 periods where $k_0 = 0.5k_{ss}$



- $k_t \to k_{ss} = 1$
- $\dfrac{k_t}{y_t} = \dfrac{K_t}{Y_t} \to 3$