ECON 8040 – TA13

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Today's Session

- PS8 due **Friday, November 17**
- Thanksgiving Break
  - No class Tuesday, November 21 nor Thursday, November 23
  - No TA session Friday, November 24
- Computation Exercise due **Thursday, November 30**
- PS9 due **Friday, December 1** at 9:00 a.m.
- Final exam **Thursday, December 7, 3:30–6:30 p.m.**

By Thanksgiving

- ⋆ Install Matlab & run VFI lecture file
- ⋆ CE (a)–(c) (no coding required)
- ⋆ PS9 Problem 1

PS7 Highlights

- No leisure preference / disutility of work ⇒ **inelastic labor supply**
- Focus on defining PP/CE & deriving solutions, not shortcuts
  - Which equation gives steady state k/n?
  - Which FOCs are used to derive it?
- Pity the grader!
  - Write clearly
  - Simplify math as much as possible

Problem 1

Infinite-horizon production economy with household and business sector

(a) Write down social planner's problem
Replacing $n_{Ht} = 1 - N_{Mt}$ and $c_{Ht} = k_{Ht}^{\alpha} n_{Ht}^{1-\alpha}$, the planner's problem is

$$\max_{\{c_{Mt}, N_{Mt}, K_{Mt+1}, k_{Ht+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log\left(\left[\mu c_{Mt}^{\rho} + (1-\mu)[k_{Ht}^{\alpha}(1-N_{Mt})^{1-\alpha}]^{\rho}\right]^{\frac{1}{\rho}} \right)$$

subject to

$$\begin{aligned} c_{Mt} + K_{Mt+1} + k_{Ht+1} &= K_{Mt}^{\alpha} N_{Mt}^{1-\alpha} + (1-\delta_K)K_{Mt} + (1-\delta_H)k_{Ht} \\ c_{Mt}, K_{Mt+1}, k_{Ht+1} &\geq 0 \\ 0 \leq N_{Mt} &\leq 1 \\ K_{M0}, k_{H0} &\text{ given} \end{aligned}$$

Problem 1

(b) Write it recursively

$$V(K_M, k_H) = \max_{\{c_M, N_M K'_M, k'_H\}} \left\{ \log\left(\left[\mu c_M^\rho + (1-\mu)[k_H^\alpha (1-N_M)^{1-\alpha}]^\rho\right]^{\frac{1}{\rho}} \right) + \beta V(K'_M, k'_H) \right\}$$

subject to

$$c_M = K_M^\alpha N_M^{1-\alpha} + (1-\delta_K)K_M + (1-\delta_H)k_H - K'_M - k'_H; \lambda$$

$$0 \leq K'_M, k'_H, K'_M + k'_H \leq K_M^\alpha N_M^{1-\alpha} + (1-\delta_K)K_M + (1-\delta_H)k_H$$

$$0 \leq N_M \leq 1$$

$$K_{M0}, k_{H0} \text{ given}$$

Problem 1

(c) Write FOC and envelope condition for Bellman
- 4 controls
- 2 states

(d) Equations that characterize steady state allocations
- c_M: aggregate feasibility
- N_M: MRS
- K_M: Euler (for business sector capital)
- K_H: Euler (for household sector capital)

Problem 2

Two countries with production sector

(b) With mobile capital, planner faces **global feasibility constraint**

$$c_t^1 + c_t^2 + k_{t+1}^1 + k_{t+1}^2 = (k_t^1)^\alpha + (k_t^2)^\alpha + (1-\delta)(k_t^1 + k_t^2); \lambda_t$$

- In the long-run, the more patient country consumes larger share of global output
- Planner allocates capital equally. Why?

$$k_{t+1}^i = \left(\frac{\alpha}{\frac{\lambda_t}{\lambda_{t+1}} - 1 + \delta} \right)^{\frac{1}{1-\alpha}}$$

Problem 3

Working with three sets of preferences $u(c, 1-n)$

(a) Recursive planning problem takes form

$$V(k) = \max_{k',n} \left\{ u(c, 1-n) + \beta V(k') \right\}$$

subject to $0 \leq k' \leq Ak^{\alpha} n^{1-\alpha} + (1-\delta)k, 0 \leq n \leq 1$

(b) Characterize steady-state

- k/n, c/n same for all 3 parts. Why?
- n varies. Why?

Problem 4

Two-sector (consumption and investment) growth model

(a) Write planner's problem and specify FOCs

- ⟶ Two sectors ⇒ two feasibility constraints
  - Y_{ct} can **only** be consumed
  - Y_{xt} can **only** be invested
- ⟶ Law of motion of capital stock can also be written as

$$K_{ct+1} + K_{xt+1} = AK_{xt} + (1-\delta)(K_{ct} + K_{xt})$$

Problem 4

(b) Write recursive planner's problem. Write FOCs and envelope conditions.

$$V(K_c, K_x) = \max_{K_c', K_x'} \left\{ \frac{(K_c^{\alpha})^{1-\sigma}}{1-\sigma} + \beta V(K_c', K_x') \right\}$$

subject to $K_c' + K_x' = AK_x + (1-\delta)(K_c + K_x); \lambda$

Problem 5

Endogenous mortality model

(a) Write planning problem

$$\max_{\{N_{t+1},c_t,h_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} N_t u(c_t)$$

subject to

$$N_t(c_t + ph_t) = N_t y \qquad \text{[Feasibility]}$$

$$N_{t+1} = \left(1 - \frac{1}{f(h_t)}\right) N_t \qquad \text{[“Law of Motion”]}$$

$$c_t, h_{t+1}, N_{t+1} \geq 0$$

$$N_0, y \text{ given}$$

Problem 5

(b) Recursive planning problem

$$V(N) = \max_{N'} \left\{ \max_{c,h} \{Nu(c)\} + V(N') \right\}$$

subject to

$$N(c + ph) = Ny$$
$$N' = \left(1 - \frac{1}{f(h)}\right) N$$
$$N', c, h \geq 0$$
$$y \text{ given}$$

⋆ Does it **discount**?

Problem 1

The Lucas Tree. (Maybe) a helpful analogy: Imagine a shipwrecked crew on an island. The only consumption good available to crew is fruit of a tree on the island. Each period t tree produces fruit d_t. Crew member with s_t shares of tree is entitled to $s_t d_t$ fruit, and shares are traded at price p_t^s. Crew can also trade one-period risk-free bonds b_t at price p_t^b.
The budget constraint is

$$c_t + p_t^b b_{t+1} + p_t^s (s_{t+1} - s_t) = b_t + s_t d_t$$

(a) Define SMCE. State *all* allocations and market clearing conditions. Write utility as $u(c)$.

Problem 1

(b) Write Recursive Competitive Equilibrium.

- ⟶ Rewrite budget constraint and use hint that $w \equiv b + s(p^s(d) + d)$
- ⟶ Allocations in (a) are *policy functions*
- ⟶ Prices in (a) are functions of dividends
- ⟶ State variables: (w, d)
- ⟶ Control variables: (c, s', b')
- ⟶ Refer to example in lecture notes

You don't need to do part (b) to solve parts (c) and (d)

Problem 1

Use SMCE definition in (a) to solve stock price p_t^s and bond price p_t^b.

(c) Assume $d_t = 1$. No utility form provided, just keep utility as $u(c)$

(d) Assume $u(c) = \log c$ and $d_t = \begin{cases} 1 & t = 0, 2, 4, \ldots \\ 2 & t = 1, 3, 5, \ldots \end{cases}$

Problem 2

Sequential markets economy w/ government

(a) Define SMCE
- ⟶ Prices, allocations, and *policy*
- ⟶ HH budget constraint includes taxes and transfers
- ⟶ Firm maximizes profit
- ⟶ Government tax revenues equal lump-sum transfers
- ⟶ Four markets clear

Problem 2

(b) Write i_{t+1} in terms of taxes, allocations

- FOC wrt a_{t+1}

(c) Write 3 equations determining **steady state** allocations

- Euler equation
- Aggregate feasibility
- MRS consumption / leisure

Problem 2

(d) Write capital-labor ratio as function of taxes, parameters
  - ⟶ Use **Euler equation**

(e) Find steady state interest rate
  - ⟶ Use equation from (b)

Problem 1

Growth Rates and Tax Policy

(a) Define Arrow-Debreu competitive equilibrium.
   - ⟼ Be careful with policy variables (τ_c, τ_k, g)

(b) Derive Euler equation of household
   - ⟼ FOCs with respect to choice variables

(c) Find long-run growth rate of the economy: $\frac{c_{t+1}}{c_t}$

(d) Can government increase growth rate by subsidizing capital? $(\tau_k < 0)$

(e) Find optimal tax subsidy that maximizes representative household's welfare.
   - ⟼ Can be shown mathematically, but trust your intuition, too.

Problem 2

(a) Write down aggregate feasibility (aggregate or per person variables $k_t \equiv \frac{K_t}{\bar{N}}$)

(b) Write down profit maximization problem. Derive formulas for rental rate of capital r_t and wage w_t

(c) Define SMCE

(d) Write 3 equations in per-person variables that characterize equilibrium allocations
- ⤚ Euler equation
- ⤚ Aggregate feasibility
- ⤚ Marginal rate of substitution consumption and leisure

(e) Find steady state capital-labor ratio $\frac{K}{H}$.

(f) How does labor income tax τ affect factor prices, capital-to-output ratio, labor supply, and total output?

Problem 3

AKH growth model with human capital accumulation

(a) Define TDCE

(b) Assume $\delta_k = \delta_h$. Find $\frac{c_{t+1}}{c_t}$ in terms of parameters and taxes.

(c) Tax reform – human capital investment x_t^h now exempt. Define TDCE.

- $\rightarrowtail$ Be careful editing government budget and household budget constraint.

(d) Find growth rates again.

(e) Holding tax rates fixed, which economy grows faster?

Problem 4

- ⋆ Definitely the hardest problem on the problem set
- ⋆ So far, we've treated population growth as an exogenous parameter. This model makes population growth *endogenous*.
- ⋆ References missing on the problem set (actually useful for solving)
  - ⟶ Barro, Robert J. and Gary S. Becker. (1989) "Fertility choice in a model of economic growth." *Econometrica* 57(2): 481–501.
  - ⟶ Becker, Gary S. and Robert J. Barro. (1988) "A reformulation of the economic theory of fertility." *Quarterly Journal of Economics* 103(1): 1–25.

Problem 4

(a) Write down aggregate feasibility
- $\rightarrowtail$ R inclusive of depreciation, so law of motion of capital has $\delta = 0$
- $\rightarrowtail$ Raising children has two forms of cost:
  - Consumption good
  - Time-spent away from work

(b) Write down planning problem of "head of family" that takes initial population N_0, capital stock K_0 as given

(c) Use planner's FOCs to solve for *per capita* consumption c_t and population growth n_t. How do they depend on N_0, K_0?

Problem 4

(d) Write planner's problem recursively. Denote value function $V(K, N)$. Show that it is a contraction mapping.

- ⟶ Blackwell's sufficient conditions

(e) Show that $V(K, N) = N^{1+\eta} v(k)$. Write Bellman equation that $v(k)$ solves.

- ⟶ $k \equiv \frac{K}{N}$
- ⟶ Plug in "guess" of $V(K, N)$ into Bellman and rearrange terms

(f) Show k' is constant and independent of k

- ⟶ It's sufficient to write k' in terms of c, n, and parameters
- ⟶ You already solved for c, n

Computation Exercise

Matlab installation instructions on eLC

(a) De-trend planner's problem (population, TFP grow)

(b) Write stationary planner's problem recursively. Determine k^{ss}:

- Write Euler equation
- Impose steady state condition: $V'(k) = V'(k')$

(c) Calibrate parameters $(A, \beta, \alpha, \delta)$

(d) Solve model **numerically by VFI**

- Lecture recording, slides, and references in eLC

(e) Use policy functions to simulate capital, consumption, and output