ECON 8040 – TA11

Michael Kotrous

University of Georgia

November 3, 2023

Today's Session

- Midterm Exam #2, PS6 grades
- PS7 due **Friday, November 10**

Midterm Exam Results

Table 1: Midterm Exams – Summary Statistics

Value	M1	M2
Mean	53	52
Max	79	87
Q_3	63	64
Q_2	53	56
Q_1	40	38

- Exam and solutions available in eLC

Problem 1

(a) Write the Bellman equation

$$V(k) = \max_{k'} \left\{ \frac{[(z+1-\delta)k - k']^{1-\sigma}}{1-\sigma} + \beta V(k') \right\}$$

subject to $0 \leq k' \leq (z+1-\delta)k$

Problem 1

(b) Solve using guess $V(k) = \frac{Ak^{1-\sigma}}{1-\sigma}$

- ⋆ First-order condition

$$k' = \frac{(z+1-\delta)k}{(\beta A)^{\frac{-1}{\sigma}} + 1}$$

- ⋆ Replace k' in $\frac{Ak^{1-\sigma}}{1-\sigma} = \frac{[(z+1-\delta)k - k']^{1-\sigma}}{1-\sigma} + \frac{\beta A k^{1-\sigma}}{1-\sigma}$

$$A = (z+1-\delta)^{1-\sigma}\beta A\left[(\beta A)^{\frac{-1}{\sigma}} + 1\right]^{\sigma}$$

- ⋆ Replace A in FOC to solve policy function:

$$\boxed{g(k) = [(z+1-\delta)\beta]^{\frac{1}{\sigma}} k}$$

Problem 1

(c) Find growth rate of k_t and c_t.

- Capital: $\frac{k'}{k} = \frac{g(k)}{k} = [(z+1-\delta)\beta]^{\frac{1}{\sigma}}$
- Consumption: $\frac{c_{t+1}}{c_t} = \frac{(z+1-\delta)k_{t+1} - g(k_{t+1})}{(z+1-\delta)k_t - g(k_t)} = \frac{\left[(z+1-\delta) - [(z+1-\delta)\beta]^{\frac{1}{\sigma}}\right]k_{t+1}}{\left[(z+1-\delta) - [(z+1-\delta)\beta]^{\frac{1}{\sigma}}\right]k_t}$

Consumption grows at the same rate as capital. The economy grows if

$$(z+1-\delta)\beta > 1$$

Problem 2

(a) FOC for optimal labor supply:

$$\frac{1-\phi}{1-n} = \frac{\phi(1-\alpha)zk^{\alpha}n^{-\alpha}}{zk^{\alpha}n^{1-\alpha} + (1-\delta)k - k'}$$

Bellman equation:

$$V(k) = \max_{k'} \left\{ F(k, k') + \beta V(k') \right\}$$

subject to

$$F(k, k') = \max_{n} \left\{ \phi \log(zk^{\alpha}n^{1-\alpha} + (1-\delta)k - k') + (1-\phi) \log(1-n) \right\}$$

$$0 \leq k' \leq zk^{\alpha}n^{1-\alpha} + (1-\delta)k$$

$$0 \leq n \leq 1$$

Problem 2

(b) Assume $\delta = 1$ and guess $V(k) = A + B \log k$. Optimality condition for k' (in terms of B and n):

$$k' = \frac{\beta B z k^{\alpha} n^{1-\alpha}}{\phi + \beta B}$$

(c) Find optimal n (in terms of B):

$$n = \frac{(1-\alpha)(\phi + \beta B)}{1 - \phi + (1-\alpha)(\phi + \beta B)}$$

(d) Replace k', n in Bellman to find B:

$$B = \frac{\alpha \phi}{1 - \alpha \beta}$$

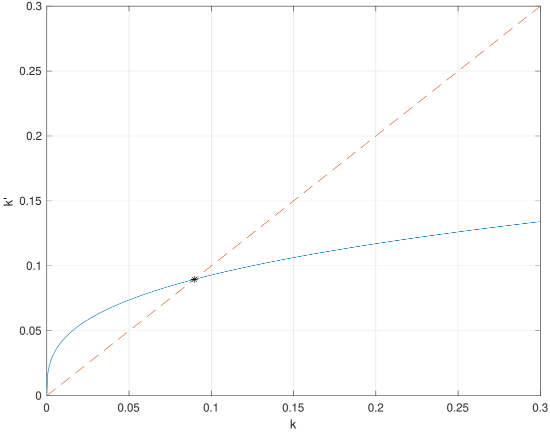
Problem 2

(e) Replace B to finish

$$n^* = \frac{\phi(1-\alpha)}{1-\alpha\beta-\alpha\phi+\alpha\beta\phi}$$

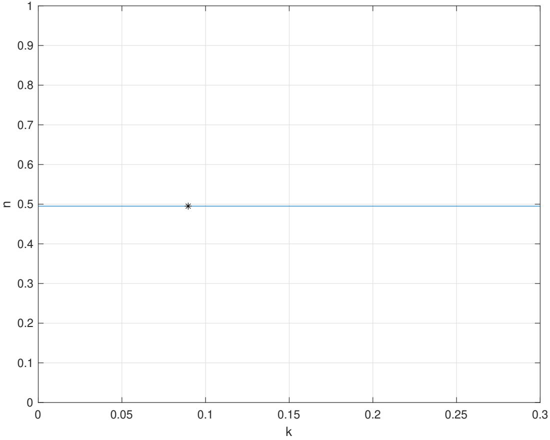
$$\Rightarrow k' = \alpha\beta z k^\alpha (n^*)^{1-\alpha} \qquad c = (1-\alpha\beta) z k^\alpha (n^*)^{1-\alpha}$$

Policy Function – Capital



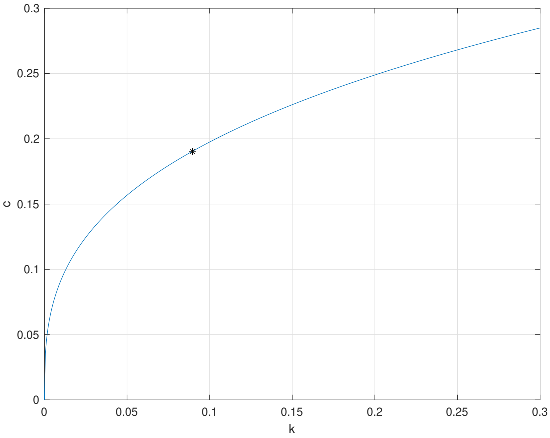
Calibration: $\alpha = 1/3$, $z = 1$, $\beta = 0.96$, $\phi = 1/2$

Policy Function – Labor



Calibration: $\alpha = 1/3$, $z = 1$, $\beta = 0.96$, $\phi = 1/2$

Policy Function – Consumption



Calibration: $\alpha = 1/3$, $z = 1$, $\beta = 0.96$, $\phi = 1/2$

Problem 3

Economy with physical and human capital accumulation

(a) Write planning problem:

$$w(k_0, h_0) = \max_{\{(c_t, k_{t+1}, h_{t+1})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

subject to

$c_t, k_{t+1}, h_{t+1} \geq 0$; $c_t + k_{t+1} + h_{t+1} \leq z k_t^{\alpha} h_t^{1-\alpha} + (1-\delta)(k_t + h_t)$

Problem 3

(b) Write planning problem recursively:

$$V(k,h) = \max_{k',h'} \left\{ \log\left[zk^{\alpha}h^{1-\alpha} + (1-\delta)(k+h) - (k'+h')\right] + \beta V(k',h') \right\}$$

subject to $0 \leq k', h', k'+h' \leq zk^{\alpha}h^{1-\alpha} + (1-\delta)(k+h)$

Problem 3

(c) Assume $\delta = 1$ and solve policy functions using guess-and-verify

- Guess $V(k,h) = D + K\log k + H\log h$
- Write FOCs for k' and h'
- Replace k' and h' in Bellman and solve for K, H
- Replace coefficients in FOCs to find policy functions:

$$k'(k,h) = \alpha\beta z k^{\alpha} h^{1-\alpha}$$
$$h'(k,h) = (1-\alpha)\beta z k^{\alpha} h^{1-\alpha}$$

Bellman has two state variables (k, h) and two control variables (k', h'), so there are *two policy functions*, each of which is a function of the two state variables!

Problem 4

(a) Write planning problem only in terms of sequence of capital $\{k_{t+1}\}_{t=0}^{\infty}$

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \theta_t \log(k_t - k_{t+1})$$

subject to $0 \leq k_{t+1} \leq k_t$, $\theta_t = \begin{cases} \theta_H & t \text{ is even} \\ \theta_L & t \text{ is odd} \end{cases}$

(b) Write two Bellman equations:

$$V(k, \theta_H) = \max_{0 \leq k' \leq k} \{\theta_H \log(k - k') + \beta V(k', \theta_L)\}$$
$$V(k, \theta_L) = \max_{0 \leq k' \leq k} \{\theta_L \log(k - k') + \beta V(k', \theta_H)\}$$

Problem 4

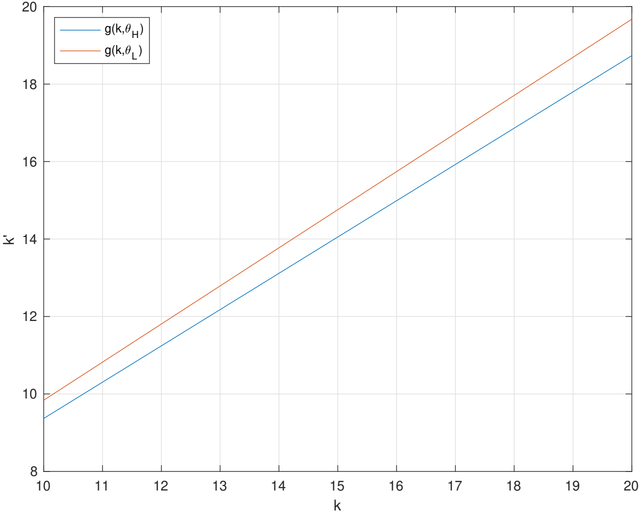
(c) Make two guesses and solve for coefficients in each

(d) Write policy functions:

$$g(k, \theta_H) = \beta \left(\frac{\theta_L + \beta\theta_H}{\theta_H + \beta\theta_L} \right) k$$
$$g(k, \theta_L) = \beta \left(\frac{\theta_H + \beta\theta_L}{\theta_L + \beta\theta_H} \right) k$$

Notice $g(k, \theta_L) > g(k, \theta_H)$

Policy Functions



Problem 1

(a) Write down planner's problem
- Carefully write down all choice variables
- Business sector aggregate feasibility: output of the business sector can be consumption of business sector good, invested in business capital, or invested in home capital
- Home-produced consumption good cannot be invested in either form of capital

(b) Write the Bellman equation
- Carefully write down state and control variables
- Replace constraints in objective where convenient
- Don't forget to write down all remaining constraints!

Problem 1

(c) Write down FOCs and envelope conditions
- If derivatives are too difficult, you may want to adjust the Bellman you wrote down in (b)

(d) Write equations that characterize the steady state
- Feasibility and first-order conditions in steady state
- There should be as many equations as "unknowns." In principle, the system can be solved.

Problem 2

Two nations vary in discount factor β_i.

(a) Write steady state capital in case of autarky (i.e., no trade). Which country has higher steady-state capital? Does the result make sense?

(b) Write planner problem when capital can cross borders.

- One feasibility constraint
- Find steady state $\frac{c_t^2}{c_t^1}$ and steady state capital in each country.

Problem 3

Given Cobb-Douglas production function, law of capital motion, and three types of preferences $u(c, 1-n)$

log-constant Frisch elasticity	$\log c - \psi \frac{n^{1+\frac{1}{\varepsilon}}}{1+\frac{1}{\varepsilon}}$
log-log	$\phi \log c + (1-\phi)\log(1-n)$
Cobb-Douglas	$\frac{\left(c^{\phi}(1-n)^{1-\phi}\right)^{1-\sigma}}{1-\sigma}$

For each:

(a) Write down recursive planning problem and FOCs, envelope conditions

(b) Characterize steady state allocations in terms of k and n

Problem 4

Economy with distinct consumption and investment goods.

(a) Write planner's problem and planner's FOCs
   - ⟼ There are two aggregate feasibility constraints on planner. It may be helpful to think of what market clearing conditions would be in a competitive equilibrium.

(b) Write planner's problem recursively and take FOCs, envelope conditions.
   - ⟼ Carefully write down state and control variables.

Problem 5

Model of endogoneous mortality from Hall and Jones (2007).

(a) Write down planner's problem
  - Assume discount factor $\beta = 1$
  - Be careful with choice variables, feasibility and law of motion

(b) Write planner's problem recursively. Does it satisfy Blackwell's sufficient conditions?
  - What is the state variable? i.e., what does planner need to know at beginning of period to make optimal allocations?
  - Write down feasibility and law of motion constraints

Problem 5

(c) Show that value function of form vN solves the Bellman. What is v?

(d) Write down FOC. What is p?

(e) Use given $u(c)$ and $f(h)$ to find health expenditure share of income, i.e., $s \equiv \frac{ph}{y}$

(f) Comparative statics: how does s change in parameters, p?